SONOPANT DANDEKAR ARTS, V.S. APTE COMMERCE AND M.H. MEHTA SCIENCE COLLEGE, PALGHAR

Department of Mathematics

PROJECT REPORT

Master of Science - Mathematics

Academic Year 2022-2023

Prepared by

Department of Mathematics

Sonopant Dandekar Arts, V.S. Apte Commerce and M.H. Mehta Science College, Palghar

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Sonopant Dandekar Arts, V.S. Apte Commerce and M.H. Mehta Science College, Palghar

Academic Year 2022 – 23

Project Presentation and Submission Notice

Date: 15/05/2023

All the students of **M.Sc II (Sem-IV)** are hereby informed that their final project presentation (10 minutes duration) and project submission will be on 25th May 2023 at 9.30 AM in BMS seminar hall.

The project presentation will be followed by the viva on the presentation.

All the students have to submit their black book and bring their project in pen drive for presentation.

Head
Department of Mathematics

(Asst. Prof. Dipali Mali)

Principal

(Dr. Kiran Save)

University of Mumbai

No. AAMS(UG)/98 of 2021-22

CIRCULAR:-

Attention of the Principals of the Affiliated Colleges, the Head of the University Departments and Directors of the Recognized Institutions in Faculties of Humanities and Science & Technology.

They are hereby informed that the recommendations made by the Board of Studies in Mathematics at its online meeting held on 23rd April, 2021 vide Item No. 1 (ii) and subsequently passed by the Board of Deans at its online meeting held on 11th June, 2021 vide item No. 6.18 (R) have been accepted by the Academic Council at its meeting held on 29th June, 2021 vide item No. 6.18(R) and that in accordance therewith, Finalize the proposed syllabus of M.Sc./M.A. (Sem-III & IV) in Mathematics under (CBCS) in Evaluation Scheme A, B & C has been brought into force with effect from the academic year 2021-22 accordingly. (The same is available on the University's website www.mu.ac.in).

MUMBAI – 400 032
8th October, 2021

(Dr. B.N.Gaikwad)
I/c REGISTRAR

To

The Principals of the Affiliated Colleges, the Head of the University Departments and Directors of the Recognized Institutions in Faculties of Humanities and Science & Technology.

A.C/6.18 (R) 29/06/2021

****

No. AAMS(UG)/ 98 -A of 2021-22 MUMBAI-400 032 8th October, 2021

Copy forwarded with Compliments for information to:-

1) The Dean, Faculties of Humanities and Science & Technology,
2) The Chairman, Board of Studies in Mathematics,
3) The Director, Board of Examinations and Evaluation,
4) The Director, Board of Students Development,
5) The Co-ordinator, University Computerization Centre,

(Dr. B.N.Gaikwad)
I/c REGISTRAR

Copy to :-

1. **The Deputy Registrar, Academic Authorities Meetings and Services (AAMS),**
2. **The Deputy Registrar, College Affiliations & Development Department (CAD),**
3. **The Deputy Registrar, (Admissions, Enrolment, Eligibility and Migration Department (AEM),**
4. **The Deputy Registrar, Research Administration & Promotion Cell (RAPC),**
5. **The Deputy Registrar, Executive Authorities Section (EA),**
6. **The Deputy Registrar, PRO, Fort, (Publication Section),**
7. **The Deputy Registrar, (Special Cell),**
8. **The Deputy Registrar, Fort/ Vidyanagari Administration Department (FAD) (VAD), Record Section,**
9. **The Director, Institute of Distance and Open Learning (IDOL Admin), Vidyanagari,**

They are requested to treat this as action taken report on the concerned resolution adopted by the Academic Council referred to in the above circular and that on separate Action Taken Report will be sent in this connection.

1. **P.A to Hon'ble Vice-Chancellor,**
2. **P.A Pro-Vice-Chancellor,**
3. **P.A to Registrar,**
4. **All Deans of all Faculties,**
5. **P.A to Finance & Account Officers, (F.& A.O),**
6. **P.A to Director, Board of Examinations and Evaluation,**
7. **P.A to Director, Innovation, Incubation and Linkages,**
8. **P.A to Director, Board of Lifelong Learning and Extension (BLLE),**
9. **The Director, Dept. of Information and Communication Technology (DICT) (CCF & UCC), Vidyanagari,**
10. **The Director of Board of Student Development,**
11. **The Director, Department of Students Walfare (DSD),**
12. **All Deputy Registrar, Examination House,**
13. **The Deputy Registrars, Finance & Accounts Section,**
14. **The Assistant Registrar, Administrative sub-Campus Thane,**
15. **The Assistant Registrar, School of Engg. & Applied Sciences, Kalyan,**
16. **The Assistant Registrar, Ratnagiri sub-centre, Ratnagiri,**
17. **The Assistant Registrar, Constituent Colleges Unit,**
18. **BUCTU,**
19. **The Receptionist,**
20. **The Telephone Operator,**
21. **The Secretary MUASA**

for information.

AC - 29/6/2021
Item No. 6.18

UNIVERSITY OF MUMBAI

Syllabus
for the
Program : M.Sc./M.A. Sem.-III & IV
CBCS
Course : Mathematics

(Choice Based and Credit System with effect from the academic year 2021-22)

AC 29/6/2021

Item No. 6.18

UNIVERSITY OF MUMBAI

Syllabus for Approval

Sr. No.	Heading	Particulars
1	Title of the Course	M. Sc. / M. A. Mathematics, Sem III& IV
2	Eligibility for Admission	As per University regulations
3	Passing Marks	40% (Internal 16/40 and External 24/60 in each Theory course and 40 marks in Project).
4	Ordinances / Regulations (if any)	-
5	No. of Years / Semesters	Two Semesters (III &IV)
6	Level	PG
7	Pattern	Semester
8	Status	Revised
9	To be implemented from Academic Year	From Academic Year : 2021-2022

Date: 19.05.2021

Signature:

19.05.2021

Name Prof. R. P. Deore

Chairman of BoS of Mathematics.

Semester IV

Sr. No.	Subject code	Units	Subject	Credits	L/W
			Algebra IV		
01	PSMT/ PAMT 401	Unit I	Algebraic Extensions	06	04
		Unit II	Normal and Separable Extensions		
		Unit III	Galois Theory		
		Unit IV	Applications		
			Fourier Analysis		
02	PSMT/ PAMT 402	Unit I	Fourier Series	05	04
		Unit II	Dirichlet's Theorem		
		Unit III	Fejer's Theorem and Applications		
		Unit IV	Dirichlet Problem in the Unit Disc		
			Calculus on Manifolds		
03	PSMT/ PAMT 403	Unit I	Multivariable Algebra	05	04
		Unit II	Differential Forms		
		Unit III	Basics of Submanifolds of R^n		
		Unit IV	Stoke's theorem		
04	PSMT/ PAMT 404	**Skill Based Course** (Any one from I to V)		04	04
05	PSMT/ PAMT 405	**Project**		04	04

Note:

- PSMT401/PAMT401, PSMT402/PAMT402, PSMT403/PAMT403 are compulsory courses for Semester IV.
- PSMT 404/PAMT 404 is Skill Based Course for Semester IV.
- Skill Based Course will be any ONE of the following list of **FIVE** courses:

1 Business Statistics

2 Statistical Methods

3 Computer Science

4 Linear and Non Linear Programming

5 Computational Algebra

4. Teaching Pattern for Semester III & IV

1. Four lectures per week for each of the courses: PSMT301/PAMT301, PSMT302/ PAMT302, PSMT303/PAMT303, PSMT304/PAMT304 and PSMT305/PAMT305.
2. Four lectures per week for each of the courses: PSMT401/PAMT401, PSMT402/ PAMT402, PSMT403/PAMT403, PSMT404/PAMT404.
3. Each lecture is of 60 minutes duration.
4. In addition, there shall be tutorials, seminars as necessary for each of the five courses.
5. PSMT 405/PAMT 405 is a project based Course for Semester IV.

The projects for this course are to be guided by the Faculty members of the Department of Mathematics of the concerned college. Each project shall have maximum of 08 (eight) students. The workload for each project is 2 L/W.

M.Sc. Part II 2022-2023

Project Details

PPT Presentation on 25th May, 2023

Sr. No.	Roll No	Name	Project Title
1	18001	Bhoye Premkumar	Riemann and lebesque integral
2	18002	Suraj Bhusara	Normed linear spaces
3	18003	Naresh Govind	Banach spaces
4	18004	Mahesh andher	Ramsey Number
5	18005	Rajesh Thapad	Special functions and fractional derivative
6	18006	Jagruti Chaudhari	Simple and Semi-simple Artinian rings
7	18007	Vivek yadav	Algebraic Expressions
8	18008	Pranali patil	Dihedral group
9	18009	Sairaj Patil	Direct and semi-direct product
10	18010	Sunita Chaware	Curves and surfaces
11	18011	Priti yadav	Shortest path problem dijkstra's algorithm
12	18012	Yajusha Marathe	Characterisation of Eulerian Graph and Randomly Eulerian Graph
13	18013	Shivam Shukla	Anova Testing
14	18014	Vaishnavi Gurada	Simple groups
15	18015	Archita Jayswal	Eulerian and hamiltonian graphs
16	18016	Sandesh Tumbada	Polya's counting theory
17	18017	Ankit Tiwari	Statistical method of sampling theory
18	18018	Rahul Mishra	Polya's counting theory
19	18019	Abhishek Pandey	Pigeonhole Principle and its applications
20	18020	Prakash yadav	Measure theory
21	18021	Pal Amit	Fixed Point Theory

Prof. Dipali Mali
Head, Mathematics Department

Dr. Kiran J. Save
Principal

PRINCIPAL
Sonopant Dandekar Arts College,
V.S. Apte Commerce College &
M.H. Mehta Science College
PALGHAR (W.R.)
Dist. Palghar, Pin-401404

M.Sc (MATHEMATICS) Internal Exam Semester IV
Academic Year 2022-23

Paper Code: PSMT405 PROJECT

Attendance Sheet for Project Submission

25/05/2023

SR.NO	ROLL NO	NAME OF THE STUDENT	SIGNATURE
1	18001	BHOYE PREMKUMAR CHIMA	Bhoye
2	18002	BHUSARA SURAJ MADHUKAR	[signature]
3	18003	GOVIND NARESH PANDU	Naresh
4	18004	ANDHER MAHESH SHANKAR	[signature]
5	18005	THAPAD RAJESH RAMESH	[signature]
6	18006	CHAUDHARI JAGRUTI SUNIL	Chaudhari
7	18007	YADAV VIVEK RAMAVTAR	Vivek
8	18008	PATIL PRANALI NAVANATH	[signature]
9	18009	PATIL SAIRAJ RAMESH	[signature]
10	18010	CHAWARE SUNITA ANANTA	Chaware
11	18011	YADAV PRITI RAMDARAS	Yadav
12	18012	MARATHE YAJUSHA CHANDRAKANT	Yajusha
13	18013	SHUKLA SHIVAM VIJAY	Shukla
14	18014	GURADA VAISHNAVI GANESH	Gurada
15	18015	JAYSAWAL ARCHITA SHIVKUMAR	Archita
16	18016	TUMBADA SANDESH LAXMAN	Sandesh
17	18017	TIWARI ANKIT KRISHNAKANT	[signature]
18	18018	MISHRA RAHUL KUMAR MITHILESH	[signature]
19	18019	PANDEY ABHISHEK SHESHDHAR	[signature]
20	18020	YADAV PRAKASH SADHUPRASAD	Prakash
21	18021	PAL AMIT RAMPREET	Amit

TOTAL STUDENTS PRESENT :- 21

TOTAL STUDENTS ABSENT :- Nil

[signature]

HEAD OF DEPARTMENT SIGN:-

(Asst. Prof. Dipali Mali)

NORMED LINEAR SPACE

A Project Submitted to the University of Mumbai

For the

M.Sc. Degree

in Mathematics

SUBMITTED BY

BHUSARA SURAJ MADHUKAR

UNDER THE GUIDANCE OF

Prof. DIPALI MALI (HOD) OF MATHEMATICS

Department of Mathematics

(University of Mumbai)

Sonopant Dandekar College,

Palghar-401404

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Sonopant Dandekar College,

Palghar-401404

May -2023

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BHUSARA SURAJ MADHUKAR

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Prof. DIPALI MALI (HOD) OF MATHEMATICS

Department of Mathematics

(University of Mumbai)

Sonopant Dandekar College,

Palghar-401404

May -2023

STATEMENT TO BE INCORPORATED BY THE CANDIDATES IN THE PROJECT AS REQUIRED UNDER REGULATION FOR THE M.Sc. DEGREE.

As required by the university Regulation No. R.1972/R.2190 we wish to state that the work embodied in this project titled "Normed Linear Space " is expository in nature and doesn't contain any new results. The entire work is carried out under the guidance of Prof Dipali Mali at the Department of mathematics, SDSM college. This work has not been submitted for any other degree of this or any other university. Whenever references have been made of previous work of others, it has been clearly indicated as such and included in the Bibliography.

Place: SDSM College, Palghar

Date: May 2023

22/05/2022

Signature of Student

Bhusara Suraj Madhukar

Certificate

This is to certify that **Bhusara Suraj Madhukar** has worked under my supervision for a Master of Science project titled **"Normed Linear Space".** The project is an expository in nature and does not contain any new results. Also, the project has not been sub- mitted to this University or any other University for any other degree.

Place: SDSM College, Palghar

Date: May 2023

22/05/2023

25/05/23

S.D.S.M. COLLEGE PALGHAR MATHEMATICS DEPT.

Signature of Guide

(Prof. Dipali Mali)

Head, Department of Mathematics,

SDSM College, Palghar-401404,

University of Mumbai, Mumbai-400 098.

Acknowledgments

I express my heartfelt indebtedness and owe a deep sense of gratitude to our professor and guide Dipali Mali Department of mathematics, SDSM College, for his guidance and support in completing my project.

I would also like to extend my gratitude to the head of our department Prof. Dipali Mali for their encouragement and for giving me this opportunity to work on this project and to the entire department of Mathematics for their co-ordination and co-operation.

I also thank all our friends and family members for supporting me through this endeavour.

Place: SDSM College, Palghar.

Date: May 2023

22/05/2023

SDSM College, Palghar-401404,

University of Mumbai,

Mumbai-400 098.

Signature of Candidates:

Bhusara Suraj Madhukar

Declaration

We declare that this written submission represents our ideas in our own words and where other's idea has been included, we have adequately cited and referenced the original sources. We also declare that we have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated are falsified any of the ideas/data/fact/source in our submission. We understand that any violation of the above will cause disciplinary action by the Institute and can also evoke penal action from the sources which thus not been properly cited or from whom proper permission has not been taken when needed.

Signature

Bhusara Suraj Madhukar

Date: May 2023

22\05\2023

Abstract:

In this paper, both the product-normed linear space P–NLS (product-Banach space) and product-semi-normed linear space (product-semi-Banch space) are introduced. These normed linear spaces are endowed with the first and second product inequalities, which have a lot of applications in linear algebra and differential equations. In addition, we showed that P–NLS admits functional properties such as completeness, continuity and the fixed point.

Key Words and Phrases: product-normed linear space, product-semi-normed linear space, completeness, fixed point theorem.

Contents

1 INTRODUCTION TO NORMED LINEAR SPACES

I stress once again that the normed linear space is the pair (X, k · k). The same vector space X can be equipped with many different norms, and these give rise to different normed linear spaces. However, we shall often refer informally to "the normed linear space X" whenever it is understood from the context what the norm is.

We saw last week that completeness is a very nice property for a metric space to have, and that every metric space can be embedded as a dense subset of a complete metric space (called its completion); indeed, if one is given an incomplete metric space, then it is advisable to first form its completion and henceforth work there.

Now, the process of forming the completion — by taking equivalence classes of Cauchy sequences — can be applied equally well to a normed linear space (which, after all, is a special case of a metric space), and this process respects the vector-space structure. Therefore, the completion of a normed linear space is again a normed linear space (or more precisely, can be given in an obvious way the structure of a normed linear space). And if one is given an incomplete normed linear space, it is advisable to first form its completion and henceforth work there.

A complete normed linear space is called a Banach space.[1] Most of the important spaces in functional analysis are Banach spaces.[2] Indeed, much of this course concerns the properties of Banach spaces.

2.1 Definition

Definition 1 - Let X be a vector space over the field R of real numbers (or the field C of complex numbers). Then a norm on X is a function that assigns to each vector $x \in X$ a real number $||x||$, satisfying the following four conditions:

(i) $||x|| \geq 0$ for all $x \in X$ (nonnegativity);

(ii) $||x|| = 0$ if and only if $x = 0$ (nondegeneracy);

(iii) $||\lambda x|| = |\lambda| \, ||x||$ for all $x \in X$ and all $\lambda \in R$ (or C) (homogeneity);

(iv) $||x + y|| \leq ||x|| + ||y||$ for all $x,y \in X$ (triangle inequality).

The pair (X, $||\cdot||$) consisting of a vector space X together with a norm $||\cdot||$ on it is called a normed linear space.

Definition 2 - A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is called a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $\alpha * 1 = \alpha$, $\forall \alpha \in [0,1]$;

(iii) $\alpha * \gamma \leq \beta * \delta$ whenever $\alpha \leq \beta$ and $\gamma \leq \delta$, $\forall \alpha,\beta,\gamma,\delta \in [0,1]$.

If $*$ is continuous then it is called continuous t-norm.

The following are examples of some t-norms.

(i) Standard intersection: $\alpha * \beta = \min\{\alpha,\beta\}$.

(ii) Algebraic product: $\alpha * \beta = \alpha\beta$.

(iii) Bounded difference: $\alpha * \beta = \max\{0,\alpha + \beta - 1\}$.

Definition 3- Let X be a linear space over a field F. A fuzzy subset N of $X \times R$ is called fuzzy norm on X if $\forall x,y \in X$ the following conditions hold:

(N1) $\forall t \in R$ with $t \leq 0$, $N(x,t) = 0$;

(N2) $(\forall t \in R, t > 0, N(x,t) = 1) \Longleftrightarrow x = \theta$;

(N3) $\forall s,t \in R$, $N(x + y,s + t) \geq N(x,s) * N(y,t)$;

(N4) $N(x,\cdot)$ is a non-decreasing function of t and

$$\lim_{t \to \infty} N(x,t) = 1.$$

Then the pair (X,N) is called fuzzy normed linear space.

Definition 4- Let (X,N) be a fuzzy normed linear space.

(i) A sequence $\{x_n\}$ is said to be convergent if $\exists x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$, $\forall t > 0$. Then x is called the limit of the sequence $\{x_n\}$ and denoted by $\lim x_n$.

(ii) A sequence $\{x_n\}$ in a fuzzy normed linear space (X,N) is said to be Cauchy if $\lim_{n \to \infty} N(x_{n+p} - x_n,t) = 1$, $\forall t > 0$ and $p = 1,2\cdots$.

(iii) $A \subseteq X$ is said to be closed if for any sequence $\{x_n\}$ in A converges to $x \in A$.

(iv) $A \subseteq X$ is said to be the closure of A, denoted by A^- if for any $x \in A^-$, there is a sequence $\{x_n\} \subseteq A$ such that $\{x_n\}$ converges to x.

(v) $A \subseteq X$ is said to be compact if any sequence $\{x_n\} \subseteq A$ has a subsequence converging to an element of A.

Definition 5 - Let (X,N) be a fuzzy normed linear space.

(i) A set $B(x,\alpha,t)$, $0 < \alpha < 1$ is defined as $B(x,\alpha,t) = \{y : N(x - y,t) > 1 - \alpha\}$.

(ii) $\tau = \{G \subseteq X : x \in G, \exists t > 0 \text{ and } 0 < \alpha < 1 \text{ such that } B(x,\alpha,t) \subset G\}$ is a topology on (X,N).

(iii) Members of τ are called open sets in (X,N).

Definition 6- A subset B of a fuzzy normed linear space (X,N) is said to be fuzzy bounded if for each r, $0 < r < 1$, $\exists t > 0$ such that $N(x,t) > 1$, $\forall x \in B$.

2.2. Completeness of spaces C(X) of bounded continuous functions

I defined the space C(X) of bounded continuous real-valued functions for a subset X of the real line. But now that we have defined continuity of maps from one metric space to another, we can see that the same definition holds when X is an arbitrary metric space. That is, we denote by C(X) the linear space of bounded continuous real-valued functions on X, and we equip it with the sup norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

Recall, also, that if A is any set, then B(A) denotes the space of bounded real-valued functions on A, equipped with the sup norm. In Problem 1 of Problem , you proved that B(A) is complete, i.e. is a Banach space.

Now, if X is any metric space, then C(X) is clearly a linear subspace of B(X), and of course the norm is the same. So to prove that C(X) is complete, it suffices to prove that C(X) is closed in B(X) (why?). Since convergence in B(X) [i.e. in the sup norm] is another name for uniform convergence of functions, what we need to prove is that the limit of a uniformly convergent sequence of bounded continuous functions on a metric space X is continuous. In your real analysis course you saw a proof of this fact when X is an interval of the real line (or a subset of R^n); the proof in the general case is identical:

Proposition 1 Let X be any metric space. Then the limit of a uniformly convergent sequence of bounded real-valued continuous functions on X is continuous.

Corollary 2 Let X be any metric space. Then the space C(X) is complete, i.e. is a Banach space.

Proof of Proposition 1. Let (f_n) be a uniformly convergent sequence of bounded real-valued continuous functions on X, and let f be the limit function; we want to prove that f is continuous. By hypothesis we have $\lim_{n\to\infty} ||f_n - f||_\infty = 0$. In other words, for any $\varrho > 0$ there exists an integer n_0 such that $||f_n - f||_\infty < \epsilon/3$ for all $n \geq n_0$. (You will see later why I chose $\epsilon/3$ here.) Now consider any point $x_0 \in X$. By continuity of f_{n0} at x_0, we can find $\delta > 0$ such that $d(x,x_0) < \delta$ implies $|f_{n0}(x) - f_{n0}(x_0)| < \epsilon/3$. But we then have

$$|f(x) - f(x_0)| \leq \quad |f(x) - f_{n0}(x)| + |f_{n0}(x) - f_{n0}(x_0)| + |f_{n0}(x_0) - f(x_0)|$$

$$< \quad \in/3 + \in/3 + \in/3$$

$$= \quad \in$$

where the first and third summands are bounded by $||f_{n0} - f||_\infty < \in/3$ (why?) and the second summand is bounded by $\in/3$ as we just saw.

This proof is often called the "$\in/3$ proof" (for obvious reasons) or the "down-over-and-up" proof (you will see why if you draw a picture).

You should give an example to show that a pointwise-convergent sequence of bounded real-valued continuous functions need not be continuous, even if the metric space X is compact (e.g. X = [0,1]) and even if the convergence is monotone (i.e. the sequence increases or decreases to its limit for each $x \in X$).

2.3 Completeness of the sequence spaces ℓ^p for $1 < p < \infty$

Let us begin by defining the ℓ^p norm on R^n (or C^n); we will then define the ℓ^p space of infinite sequences.

Example 1- R^n **with the** ℓ^p **norm.** Let p be any positive real number. Then, for any n-tuple $x = (x_1, x_2, \ldots, x_n)$ of real (or complex) numbers, we define

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} .$$

It is easy to see that this is a norm on R^n (or C^n) when p = 1; we proved it also for p = 2 using the Cauchy–Schwarz inequality. Now we will do it for all p in the interval $(1,\infty)$ by using H¨older's inequality, which generalizes the Cauchy–Schwarz inequality (and reduces to it when p = 2). The proof will use the following sequence of inequalities:

Lemma 3 - (Arithmetic-geometric mean inequality) Let x,y > 0 and $0 < \lambda < 1$. Then

$$x^\lambda y^{1-\lambda} \leq \lambda x + (1 - \lambda) y .$$

Corollary 4 - (Young's inequality) Let $p \in (1,\infty)$, and define $q \in (1,\infty)$ *by* $\frac{1}{p} + \frac{1}{q} = 1$.

Then, for any real numbers $\alpha, \beta > 0$, we have

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} .$$

The relation $\frac{1}{p} + \frac{1}{q} = 1$ will come up over and over again in discussing the ℓ

refer to q as the conjugate index to p.

Proposition 5-Consider once again $p,q \in (1,\infty)$ satisfying

$$\frac{1}{p}+\frac{1}{q}=1.$$

Then, for any $x = (x_1,x_2,...,x_n)$ and $y = (y_1,y_2,...,y_n)$ in R^n (or C^n), we have

$$\sum_{i=1}^{n}|x_i y_i| \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|y_i|^q\right)^{1/q},$$

or in other words

$$\sum_{i=1}^{n}|x_i y_i| \le \|x\|_p \, \|y\|_q .$$

Note that when $p = q = 2$ this is the Cauchy–Schwarz inequality. Note also that (6) also holds when $p = 1$ and $q = \infty$ (or the reverse) when we interpret $||\cdot||_\infty$ as the sup norm.[3]

Proposition 6- Let $1 \le p < \infty$. Then, for any $x = (x_1,x_2,...,x_n)$ and $y = (y_1,y_2,...,y_n)$ in R^n (or C^n), we have

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p},$$

or in other words

$$||x+y||_p \le ||x||_p + ||y||_p .$$

This is the needed triangle inequality for R^n (or C^n) with the ℓ^p norm.

Proof of Arithmetic-geometric mean inequality. Since the exponential function is convex (why?), we have

$x^\lambda y^{1-\lambda} = \exp[\lambda\log x + (1-\lambda)\log y] \le \lambda\exp[\log x] + (1-\lambda)\exp[\log y] = \lambda x + (1-\lambda)y$.

Proof of Young's inequality. Just put $\lambda = 1/p$, $x = \alpha^p$ and $y = \beta^q$ in the arithmetic geometric mean inequality.

Proof of Holder's inequality.¨ The result is trivial if $x = 0$ or $y = 0$ or both, so let x,y be nonzero vectors in R^n or C^n. For each index i ($1 \le i \le n$), put

$$\alpha = \frac{|x_i|}{\|x\|_p} \quad \text{and} \quad \beta = \frac{|y_i|}{\|y\|_q}$$

into the inequality of Corollary 5 and then sum over i. We get

$$\frac{\sum_{i=1}^{n}|x_i y_i|}{\|x\|_p\,\|y\|_q} \le \frac{1}{p} + \frac{1}{q} = 1.$$

Proof of Minkowski's inequality. It is trivial when p = 1, so consider $1 < p < \infty$. We have

$$\sum_{i=1}^{n}|x_i+y_i|^p \le \sum_{i=1}^{n}|x_i|\,|x_i+y_i|^{p-1} + \sum_{i=1}^{n}|y_i|\,|x_i+y_i|^{p-1}.$$

Now use H¨older's inequality to bound each of the two sums on the right: we have

X|xi| |xi + yi|p−1

i=1

$$\le \left(\sum_{i=1}|x_i|^p\right) \left(\sum_{i=1}|x_i+y_i|^{(p-1)q}\right.!$$

$$= \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/q}$$

since (p − 1)q = p, and similarly

$$\sum_{i=1}^{n}|y_i|\,|x_i+y_i|^{p-1} \le \left(\sum_{i=1}^{n}|y_i|^p\right)^{1/p} \left(\sum_{i=1}^{n}|x_i+y_i|^p\right)^{1/q}.$$

We therefore have

$$\sum^{n}|x_i+y_i|^p \le (\|x\|_p + \|y\|_p)\left(\sum^{n}|x_i+y_i|^p\right)^{1/q}.$$

i=1 i=1

$$\left(\sum_{i=1}^{n}|x_i+y_i|^p\right.$$ 1/q

If x + y 6= 0 we can divide both sides by! to obtain the desired inequality

(since 1 − 1/q = 1/p); and if x + y = 0 the desired inequality is trivial.

Example 2 - The sequence space ℓ^p. For any sequence $x = (x_1, x_2, \ldots)$ of real (or complex) numbers and any real number $p \in [1,\infty)$, we define

$$\|x\|_p = \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{1/p}.$$

Of course, $||x||_p$ defined in this way might be $+\infty$! We define ℓ^p to be the set of sequences for which $||x||_p < \infty$. We must now verify that

(a) ℓ^p is a linear space, i.e. $x,y \in \ell^p$ imply $x + y \in \ell^p$; and

or in other words

$$||x+y||_p \leq ||x||_p + ||y||_p .$$

Proof. From the finite-dimensional Minkowski's inequality we have for every n

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$$

$$\leq \left(\sum^{\infty} |x_i|^p\right)^{1/p} + \left(\sum^{\infty} |y_i|^p\right)^{1/p},$$

(b) $||\cdot||_p$ is a norm on ℓ^p.

As always, the only nontrivial thing to be verified in (b) is the triangle inequality $||x+y||_p \leq ||x||_p + ||y||_p$; and this will also demonstrate (a).

Proposition 7 - Let $1 \leq p < \infty$. Then, for any $x = (x_1,x_2,...)$ and $y = (y_1,y_2,...)$ in ℓ^p, we have $x + y \in \ell^p$ and

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

which is $< \infty$ since by hypothesis $x,y \in \ell^p$. We can now take $n \to \infty$ to conclude that $\sum_{i=1}^{\infty} |x_i + y_i|^p < \infty$ and that holds.

So we have shown that ℓ^p is a normed linear space for each $p \in [1,\infty)$. The next step is to show that ℓ^p is complete, hence is a Banach space. The proof is given in Kreyszig, Section is essentially the same for general $p \in [1,\infty)$ as it is for $p = 1$.

2.4.Elementary properties of normed linear spaces

Let $(X, ||\cdot||)$ be a normed linear space. The following properties are easy consequences of the triangle inequality, and you should supply the proofs:

Lemma 8 - Let (X, || · ||) be any normed linear space. Then:

(a) For any x,y ∈ X, we have $\left|\|x\| - \|y\|\right| \le \|x - y\|$.
In particular, the norm i
Is continues function on x.

||λ||+ (1 − λ)y|| ≤ λ||x||+ (1 − λ) ||y||, i.e. the norm is a convex function on X. In particular, open and closed balls are convex sets in

X, $\underline{i.e.}$ $x, y \in B(x_0, r)$

imply λx + (1 − λ)y ∈ B(x_0,r) for all 0 ≤ λ ≤ 1, and likewise for B(x_0,r).

(c) The vector-space operations are jointly continuous: i.e. if $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$; and if $x_n \to x$ and $\lambda_n \to \lambda$, then $\lambda_n x_n \to \lambda x$.

3.1. Subspaces and quotient spaces

Whenever one introduces a new mathematical structure, it is natural to seek operations that construct new examples of that structure out of old ones. In linear algebra we are familiar with two elementary ways of getting new vector spaces out of old ones: namely, linear subspaces and

quotient spaces. Here we want to investigate how these constructions work in the context of a normed linear space.

Subspaces are easy: Let $(X, ||\cdot||)$ be any normed linear space, and let M be any linear subspace of X. Then the restriction of $||\cdot||$ to M is obviously a norm on M, let us call it $||\cdot||_M$. So $(M, ||\cdot||_M)$ is a normed linear space; it is called the normed linear subspace of X corresponding to the linear subspace M.

Usually we will just refer to "the subspace M of X" rather than the more pedantic "the subspace $(M, ||\cdot||_M)$ of $(X, ||\cdot||)$" — the norm being understood from the context. We then have the following fundamental result:

Proposition 1- Let X be a Banach space (i.e. a complete normed linear space) and let M be a closed linear subspace of X. Then M is a Banach space (i.e. is complete).

Proof. This is an immediate consequence of the elementary fact that a closed subspace of a complete metric space is complete. (The fact that M is a linear subspace of X is irrelevant to the proof that M is complete.)

We also have the following easy fact:

Proposition 2- Let X be a normed linear space, and let M be a linear subspace of X.

Then the closure $\overline{M}$ is also a linear subspace of X.

This is an easy consequence of the joint continuity of algebraic operations and you should supply the details of the proof yourself.

The second, and slightly more complicated, construction is that of a quotient space. Let us first recall how quotient spaces are constructed in linear algebra. Let X be any vector space and let M be any linear subspace of X. Then the quotient space X/M is defined to be the linear space of cosets

$$[x] = x + M = \{x + y: y \in M\}$$

under the algebraic operations

$$[x] + [y] \equiv (x + M) + (y + M) = (x + y) + M = [x + y] \qquad \text{for } [x],[y] \in X/M$$

and

$$\lambda[x] \equiv \lambda(x + M) = \lambda x + M = [\lambda x] \qquad \text{for } [x] \in X/M \text{ and } \lambda \in R \text{ (or } C).$$

The coset [0] is the zero vector of X/M and is simply denoted 0. It is routine to verify that all the axioms of a vector space are satisfied.

Now let us equip the quotient space with a norm. A little thought shows that the right definition is $||[x]||_{X/M} = d(0, x + M) = \inf_{y \in x+M} ||y|| = \inf_{m \in M} ||x + m||$.

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We call $||\cdot||_{X/M}$ the quotient norm on the space X/M that is induced by the norm $||\cdot||$ on X. Of course, we still need to verify that this is a norm; and one slight subtlety arises in this verification:

Proposition 3 - $||\cdot||_{X/M}$ is a norm on X/M if and only if M is a closed linear subspace of X.

Proof. It is routine to verify properties (i), (iii) and (iv) of the norm $||\cdot||_{X/M}$, and I leave it as an exercise for you (in particular, you should make sure you know how to prove the triangle inequality). The property that can go wrong is the nondegeneracy property (ii), in

case M is not closed. Indeed, if M is not closed, we can take any $x \in \overline{M} \setminus M$, and then $||[x]||_{X/M} = 0$ (why?) but $[x] = [0]$ (6 why?). On the other hand, if M is closed, then so is $x + M$ (why?), and $d(0, x + M) = 0$ if and only if $0 \in x + M$ by Proposition 1.14 of

, i.e. if and only if $[x] = [0]$.

The next step is to verify completeness:

Proposition 4 -Let X be a Banach space (i.e. a complete normed linear space) and let M be a closed linear subspace of X. Then X/M is a Banach space (i.e. is complete).

Proof. Consider a Cauchy sequence $\{x_n + M\}$ in $(X/M, ||\cdot||_{X/M})$. Then for each positive integer k there exists an integer n_k such that

$$||(x_m + M) - (x_n + M)||_{X/M} < 2^{-k} \qquad \text{for all } m,n \geq n_k$$

and we can also choose the n_k so that $n_1 < n_2 < \dots$. Now consider the subsequence

$\{x_{n_k} + M\}$. We have

$$||(x_{n_k} + M) - (x_{n_{k+1}} + M)||_{X/M} < 2^{-k}.$$

By definition of the quotient norm, it follows that we can successively choose $y_k \in x_{n_k} + M$ such that

$$||y_k - y_{k+1}|| < 2^{-k}.$$

But it follows from this that $\{y_k\}$ is a Cauchy sequence in X (why? do you see why we used 2^{-k} instead of e.g. $1/k$?). Therefore, since X is complete, $\{y_k\}$ converges to some $y \in X$.

But then

$||(x_{n_k}+M)-(y+M)||_{X/M} = ||(y_k+M)-(y+M)||_{X/M} \le ||y_k-y|| \to 0$ as $k \to \infty$

so $\{x_{n_k} + M\}$ is convergent to $y + M$ in $(X/M, ||\cdot||_{X/M})$. But $\{x_{n_k} + M\}$ is a convergent subsequence of the original Cauchy sequence $\{x_n + M\}$, so $\{x_n + M\}$ also converges to $y+M$

Very soon we will introduce a third operation for constructing new normed linear spaces out of old ones: namely, forming spaces of linear mappings.

3.2. Continuous linear mappings

In linear algebra you studied linear mappings T from one vector space X to another vector space Y. (Also called linear operators — the two terms are synonyms.) Here we would like to see how linear mappings fit together with the normed structure.

So let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed linear spaces, and let $T: X \to Y$ be a linear mapping. Let us define

$$\|T\|_{X\to Y} \equiv \sup_{\substack{x\in X \\ \|x\|_X = 1}} \|Tx\|_Y = \sup_{\substack{x\in X \\ \|x\|_X \le 1}} \|Tx\|_Y = \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} .$$

(You should make sure you understand why all these definitions are equivalent.) Of course, $||T||_{X\to Y}$ might be $+\infty$. We say that the map T is bounded if $||T||_{X\to Y} < \infty$.[3]

Remark. Another equivalent definition is

$$||T||_{X\to Y} = \inf\{M: ||Tx||_Y \le M||x||_X \text{ for all } x \in X\}$$

provided that the infimum of an empty set is defined to be $+\infty$. You should supply the details of the proof that this is indeed equivalent to the other definitions.

We then have the following fundamental fact:

Proposition 5 - Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed linear spaces, and let $T: X \to Y$ be a linear mapping. Then the following are equivalent:

(a) T is continuous at 0.

(b) T is continuous.

(c) T is uniformly continuous.

(d) T is bounded.

Proof. The equivalence of (a), (b) and (c) is an easy consequence of the properties of the norm and the linearity of T (you should work out the details for yourself). Furthermore, if T is bounded, then it is clearly continuous at 0 (why?). On the other hand, suppose that T is not bounded; then for each positive integer n there exists $x_n \in X$ with $||x_n||_X = 1$ and $||Tx_n||_Y \geq n$. We then have $x_n/n \to 0$ in X but $T(x_n/n) \to6 0$ in Y since $||T(x_n/n)||_Y \geq 1$. This shows that T is not continuous at 0.

In the special case where T is a bijection, the inverse map $T^{-1}: Y \to X$ is well-defined, and we have the following criterion for its continuity:

Proposition 6 - Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed linear spaces, and let $T: X \to Y$ be a bijective linear mapping. Then the following are equivalent:

(a) T^{-1} is continuous at 0.

(b) T^{-1} is continuous.

(c) T^{-1} is uniformly continuous.

(d) T^{-1} is bounded.

(e) There exists $m > 0$ such that $||Tx||_Y \geq m||x||_X$ for all $x \in X$.

Proof. The equivalence of (a)–(d) is just Proposition 3.14 applied to T^{-1}. On the other hand, (e) is equivalent to the statement that $||T^{-1}y||_X \leq m^{-1}||y||_Y$ for all $y \in Y$, which is precisely the statement that T^{-1} is bounded (by m^{-1}).

A bijective linear mapping $T: X \to Y$ is called a topological isomorphism (or linear homeomorphism) if both T and T^{-1} are continuous.

Corollary 7 - Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed linear spaces, and let $T: X \to Y$ be a bijective linear mapping. Then T is a topological isomorphism if and only if there exist numbers $m > 0$ and $M < \infty$ such that

$$m||x||_X \leq ||Tx||_Y \leq M||x||_X$$

for all $x \in X$.

Of course, if $T: X \to Y$ is a topological isomorphism, then so is $T^{-1}: Y \to X$; and if $S: X \to Y$ and $T: Y \to Z$ are topological isomorphisms, then so is $T \circ S: X \to Z$ (why?).

Two normed linear spaces X and Y are said to be topologically isomorphic if there exists a topological isomorphism $T: X \to Y$. From what has just been said, topological isomorphism is an equivalence relation among normed linear spaces (i.e. is reflexive, symmetric and transitive).

It is worth observing that completeness of normed linear spaces is invariant under topological isomorphism:

Proposition 8 - Suppose that the normed linear spaces X and Y are topologically isomorphic. If X is complete, then so is Y.

Proof. Let $T: X \to Y$ be a topological isomorphism. If (y_n) is a Cauchy sequence in Y, then $(T^{-1}y_n)$ is a Cauchy sequence in X (why?). Since X is complete, there exists $x \in X$ such that $T^{-1}y_n \to x$ in X. But then $y_n \to Tx$ in Y (why?).

Do you see how the boundedness of both T and T^{-1} was used in this proof?

Two norms $||\cdot||_1$ and $||\cdot||_2$ on the same space X are said to be equivalent if the identity mapping id: $(X, ||\cdot||_1) \to (X ||\cdot||_2)$ is a topological isomorphism. By Corollary we see that $||\cdot||_1$ and $||\cdot||_2$ are equivalent if and only if there exist numbers $m > 0$ and $M < \infty$ such that

$$M||x||_1 \leq ||x||_2 \leq M||x||_1.$$

(This property is sometimes taken as the definition of equivalent norms, and then the topological-isomorphism property is proven to be an equivalent characterization.) Of course, if norms $||\cdot||_1$ and $||\cdot||_2$ are equivalent and norms $||\cdot||_2$ and $||\cdot||_3$ are equivalent, then norms $||\cdot||_1$ and $||\cdot||_3$ are equivalent; so equivalence of norms is an equivalence relation.

If normed linear spaces X and Y are topologically isomorphic, then they are essentially identical for all topological purposes (i.e. things relating to open sets, convergence, continuity, etc.) but not necessarily for metric purposes (i.e. things relating to distances, balls, etc.). A stronger notion is isometric isomorphism: a bijective linear mapping $T: X \to Y$ is called an isometric isomorphism if $||Tx||_Y = ||x||_X$ for all $x \in X$, i.e. if we can take $m = M = 1$ in . Two normed linear spaces X and Y are said to be isometrically isomorphic if there exists an isometric isomorphism $T: X \to Y$. If normed linear spaces X and Y are isometrically isomorphic, then they are essentially identical for all purposes.

3.3. Spaces of continuous linear mappings

At the purely algebraic level, we can define **spaces of linear mappings** as follows: If X and Y are vector spaces over the same field (either R or C), then the space L(X,Y) of all linear mappings from X to Y is a vector space under the obvious definitions of pointwise addition and multiplication by a scalar.

Now let X and Y be normed linear spaces (again over the same field); we define B(X,Y) to be the space of all continuous (or equivalently, bounded, hence the choice of the letter B)

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[5]If C is a convex subset of a real vector space V , a point $x \in C$ is called an extreme point of C if there do not exist distinct points $y,z \in C$ and a number $0 < \lambda < 1$ such that $x = \lambda y + (1 - \lambda)z$.

linear mappings from X to Y .[4] It is obviously a linear subspace of L(X,Y) (why?). More importantly, it is a normed linear space under the **operator norm** $||\cdot||_{X\to Y}$:

Proposition 9 - Let X and Y be normed linear spaces, with Y complete (i.e. a Banach space). Then B(X,Y) is complete (i.e. is a Banach space).

Proof. Let (T_n) be a Cauchy sequence in B(X,Y). We aim to show that (T_n) converges to some $T \in B(X,Y)$. The structure of the proof is the same as in our previous completeness proofs: First we identify the putative limit T, by using convergence in a weaker sense than convergence in the norm of B(X,Y); next we prove that T belongs to the space B(X,Y); and finally, we prove that (T_n) actually converges to T in the norm of B(X,Y).

Step 1. Since (T_n) is a Cauchy sequence in B(X,Y), it follows that $(T_n x)$ is a Cauchy sequence in Y for each $x \in X$ (why?). Since Y is assumed complete, this means that $(T_n x)$ converges to some (necessarily unique) element of Y , call it Tx. The mapping $T: X \to Y$ defined by x 7→ Tx is linear (why?).

Step 2. Since (T_n) is a Cauchy sequence in B(X,Y), it is necessarily bounded, i.e. there exists M such that $||T_n||_{X\to Y} \le M$ for all n. It easily follows from this that $||T||_{X\to Y} \le M$ (why?). So T is indeed a bounded linear operator from X to Y , i.e. it belongs to the space B(X,Y).

Step 3. Fix $\varrho > 0$. Since (T_n) is a Cauchy sequence in B(X,Y), there exists n_0 such that

$$||T_m - T_n||_{X\to Y} \le \varrho \text{ whenever } m,n \ge n_0,$$

and hence that

$$||T_m x - T_n x||_Y \le \in ||x||_X \text{ for all } x \in X \text{ whenever } m,n \ge n_0.$$

Now let $n \to \infty$ with m fixed; we have $T_n x \to Tx$ by construction, and hence

$$||T_m x - Tx||_Y \le \varrho ||x||_X \text{ for all } x \in X \text{ whenever } m \ge n_0$$

$$||T_m - T||_{X\to Y} \le \in \text{ whenever } m \ge n_0,$$

which is exactly what is needed to prove that $T_m \to T$ in the space B(X,Y).

3.4. Some examples and counterexamples

Example 1: Linear mappings from R^n to R^m. As you undoubtedly recall from linear algebra, any m × n matrix $A = (a_{ij})$ of real numbers defines a linear mapping (by abuse of language let us also call it A) from R^n to R^m by

$$(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for } i = 1,2,\dots,m.$$

And conversely, every linear mapping from R^n to R^m is induced by such a matrix (why?).

Let us now show that every such mapping is bounded. To give meaning to this statement, we first have to equip R^n and R^m with norms. We have already studied three such norms — the ℓ^1, ℓ^2 and ℓ^∞ norms — and it is easy to see that these three norms are all equivalent on any space R^n (why? note that this statement does not survive if we try to take $n \to \infty$: why?). So it doesn't matter which one we choose; let's choose for simplicity the ℓ^∞ norm for both the domain space R^n and the range space R^m. I then claim that the operator norm $||A||_{(Rn,||\cdot||\infty)\to(Rm,||\cdot||\infty)}$ for such a mapping is given in terms of the matrix $A = (a_{ij})$ by

$$\|A\|_{(\mathbb{R}^n,\|\cdot\|_\infty)\to(\mathbb{R}^m,\|\cdot\|_\infty)} \;=\; \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| \quad .$$

I leave it to you to prove this . It follows, in particular, that every linear mapping from R^n to R^m is bounded.

The same considerations apply if A is an m ×n matrix of complex numbers, and lead to a bounded linear mapping from C^n to C^m. The formula for the norm is the same.

We will see later, in fact, that every linear mapping from a finite-dimensional normed linear space to any normed linear space (finite-dimensional or not) is bounded .

Example 2: Linear mappings on sequence spaces induced by infinite matrices. In a similar way, we can start from an "infinite matrix" $A = (a_{ij})^{\infty}_{i,j=1}$ of real numbers, and try to define a linear operator on a sequence space (for example, on one of our spaces ℓ^p) by

$$(Ax)_i \;=\; \sum_{j=1}^{\infty} a_{ij} x_j \qquad \text{for } i = 1,2,\ldots .$$

But now things become nontrivial: we have to verify, first of all, that the sum is indeed convergent for all x in the domain space, and secondly, that the resulting sequence Ax indeed belongs to the desired range space. Only if we can do this does it then make sense to inquire whether the resulting linear map is bounded or not.

I won't try to do this in general, but will simply give a few important examples:

Consider the infinite matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix},$$

whose matrix elements are

$$a_{ij} = \begin{cases} 1 & \text{if } j - i = 1 \\ 0 & \text{otherwise} \end{cases}$$

This matrix can be interpreted as acting on any of the ℓ^p spaces ($1 \le p \le \infty$) and gives

$$A(x_1,x_2,x_3,\ldots) = (x_2,x_3,x_4,\ldots)$$

(why?). You should convince yourself that $x \in \ell^p$ implies $Ax \in \ell^p$ and that $\|A\|_{\ell^p\to\ell^p} = 1$. This operator is called **left shift**.

Similarly, one can define the **right shift** on ℓ^p: you should figure out what is the corresponding infinite matrix B and show that again $\|B\|_{\ell^p\to\ell^p} = 1$.

There are also **weighted shifts**, in which the infinite sequence of 1's in the matrix of A (or B) is replaced by a suitable sequence $(a_i)_{i=1}^{\infty}$ of scalars, so that, for instance, the weighted left shift sends $(x_1,x_2,x_3,\ldots)$ to $(a_1x_2,a_2x_3,a_3x_4,\ldots)$. Of course, we have to be careful in our choice of weights $(a_i)_{i=1}^{\infty}$ if we want to make sure that the domain space is mapped into the desired range space and that the mapping is bounded. Can you see that if the sequence $(a_i)_{i=1}^{\infty}$ is bounded, then the weighted left (or right) shift defines a bounded linear map from any ℓ^p space to itself? The class of weighted shift operators on ℓ^2 plays an important role in the theory of linear operators on Hilbert spaces.

Some more examples of linear maps defined by infinite matrices are given in Problem 4(c,d) .

Example 3: Integral operators. We consider the space C[a,b] of (automatically bounded) continuous functions on a bounded closed interval $[a,b] \subset R$, equipped with the sup norm; we will define some linear mappings from C[a,b] to itself via integrals. To do this, let $K: [a,b]\times[a,b] \to R$ be a continuous function (called the **kernel**), and define a mapping $T: C[a,b] \to C[a,b]$ by

$$(Tf)(s) = \int_a^b K(s,t)\, f(t)\, dt\ .$$

It is not hard to show that Tf is continuous whenever f is, and that T defines a bounded linear map from C[a,b] to itself, of norm

$$\|T\|_{C[a,b]\to C[a,b]} = \sup_{s\in[a,b]} \int_a^b |K(s,t)|\, dt$$

Do you see that this is an infinite-dimensional version of Example 1, in which the matrix (a_{ij}) is replaced by the kernel K(s,t) and the sum over j is replaced by integration over t? And do you see that (3.48) is the analogue of (3.42)? The operator T defined by is called a Fredholm integral operator of the first kind.[5]

If T is a Fredholm integral operator of the first kind, then U = I – T (where I denotes the identity operator) is called a Fredholm integral operator of the second kind: explicitly it is given by

$$(Uf)(s) = f(s) - \int_a^b K(s,t)\, f(t)\, dt$$

If in (3.47) or (3.49), we change the upper limit of integration from b (a fixed limit) to s (a variable limit), then T or U is called a Volterra integral operator (of the first or second kind, respectively).[6]

Now let us look at some examples of what can go wrong in infinite dimensions:

Example 4: An unbounded linear map. Let A be the infinite diagonal matrix diag(1,2,3,...), and let us try to define the corresponding linear operator A on (for example) the sequence space ℓ^2, i.e.

$$(Ax)_k = k\, x_k\,.$$

The trouble is that this formula does not map ℓ^2 into itself — can you see why? (You should give an example of an $x \in \ell^2$ with the property that $Ax \notin \ell^2$.) Indeed, the largest space on which the operator A can be sensibly defined (as a mapping into ℓ^2) is the subspace M (ℓ^2 given by

$$M = \{x \in \ell^2 \colon \sum_{k=1}^{\infty} k^2\, |x_k|^2 < \infty\}$$

It is not hard to see that M is indeed a linear subspace of ℓ^2, but it is not closed; indeed, it is a proper dense subspace of ℓ^2. (Can you prove this? Indeed, you might as well go farther and show that the subspace c_{00} (M consisting of sequences with at most finitely many nonzero entries is already dense in ℓ^2.)

So now consider A as a linear map from M (equipped with the ℓ^2 norm that it inherits as a subspace of ℓ^2) into ℓ^2. Is it bounded? The answer is no: consider the vector e_n having a 1 in the nth coordinate and 0 elsewhere; it belongs to M (and indeed to c_{00}) and we have $\|e_n\|_{\ell 2} = 1$ but $\|Ae_n\|_{\ell 2} = n$ (why?); so the operator norm of A is infinite.

This is a typical situation: Unbounded operators cannot be defined in any sensible way on the whole Banach space, but only on a proper dense subspace.

Here is another example of the same phenomenon:

Example 5: A differential operator. In the space C[0,1] of (bounded) continuous functions on the interval [0,1], equipped with the sup norm, let us try to define the operator D of differentiation. The trouble, once again, is that differentiation makes no sense for arbitrary functions $f \in C[0,1]$ (at least, not if we want the derivative f' to also belong to C[0,1]). Rather, we need to restrict ourselves to (for example) the space $C^1[0,1]$ consisting of functions that are once continuously differentiable on [0,1] (where we insist on having also one-sided derivatives at the two endpoints, and the derivative f' is supposed to be continuous on all of [0,1], including at the endpoints). This is a linear

$$(Uf)(s) = f(s) - \int_a^b K(s,t)\, f(t)\, dt$$

If in (3.47) or (3.49), we change the upper limit of integration from b (a fixed limit) to s (a variable limit), then T or U is called a Volterra integral operator (of the first or second kind, respectively).[6]

Now let us look at some examples of what can go wrong in infinite dimensions:

Example 4: An unbounded linear map. Let A be the infinite diagonal matrix diag(1,2,3,...), and let us try to define the corresponding linear operator A on (for example) the sequence space ℓ^2, i.e.

$$(Ax)_k = k\, x_k .$$

The trouble is that this formula does not map ℓ^2 into itself — can you see why? (You should give an example of an $x \in \ell^2$ with the property that $Ax \notin \ell^2$.) Indeed, the largest space on which the operator A can be sensibly defined (as a mapping into ℓ^2) is the subspace $M \subset \ell^2$ given by

$$M = \{x \in \ell^2 : \sum_{k=1}^{\infty} k^2 |x_k|^2 < \infty\}$$

It is not hard to see that M is indeed a linear subspace of ℓ^2, but it is not closed; indeed, it is a proper dense subspace of ℓ^2. (Can you prove this? Indeed, you might as well go farther and show that the subspace $c_{00} \subset M$ consisting of sequences with at most finitely many nonzero entries is already dense in ℓ^2.)

So now consider A as a linear map from M (equipped with the ℓ^2 norm that it inherits as a subspace of ℓ^2) into ℓ^2. Is it bounded? The answer is no: consider the vector e_n having a 1 in the nth coordinate and 0 elsewhere; it belongs to M (and indeed to c_{00}) and we have $||e_n||_{\ell 2} = 1$ but $||Ae_n||_{\ell 2} = n$ (why?); so the operator norm of A is infinite.

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Example 6: A bounded linear bijection with unbounded inverse. Consider the linear map T: $\ell^2 \to \ell^2$ defined by

$$(Tx)_j = \frac{1}{j} x_j \,.$$

Can you see that T is bounded? What is its operator norm?

Can you also see that T is an injection? It is not a surjection, i.e. its image is not all of ℓ^2; rather, its image is precisely the dense linear subspace M (ℓ^2 defined in Example 4 above — can you see why?

Since T is a linear bijection from ℓ^2 to M, the inverse map T^{-1} is well-defined as a map from M to ℓ^2; it is obviously given by

$$(T^{-1}x)_j = j\, x_j \,,$$

which is nothing other than the unbounded linear map A: M → ℓ^2 discussed in Example 4 above. So a bounded linear bijection T: ℓ^2 → M can have an unbounded inverse.

This pathology arises from the fact that the space M is not complete. We will see later in this course that a bounded linear bijection from a Banach space onto another Banach space always has a bounded inverse: this is a corollary of one of the fundamental theorems of functional analysis, the Open Mapping Theorem.

This same map T could have been considered in any of the spaces ℓ^p $(1 \le p \le \infty)$, not just p = 2, with similar results; I leave it to you to work out the details.

4.1. Special properties of finite-dimensional spaces

In this section we would like to study some special properties possessed by finite-dimensional normed linear spaces — properties that do not hold in general for infinite-dimensional normed linear spaces. The culmination of this section is F. Riesz's theorem stating that the closed unit ball of a normed linear space is compact if and only if the space is finite-dimensional.

Proposition 4.1 - Let $(X, ||\cdot||)$ be a finite-dimensional normed linear space over the real (resp. complex) field, and let $\{a_1,\ldots,a_n\}$ be any basis for X. Then the mapping

$$T\colon (\xi_1,\ldots,\xi_n) \mapsto \sum_{i=1}^{n} \xi_i a_i$$

of $(\mathbb{R}^n, \|\cdot\|_\infty)$ [resp. $(\mathbb{C}^n, \|\cdot\|_\infty)$] onto $(X, ||\cdot||)$ is a topological isomorphism.

Proof. I shall give the proof for R^n; it is identical for C^n.

Clearly T is a linear bijection from R^n to X (why?). Moreover, for $\xi = (\xi_1,\ldots,\xi_n) \in R^n$ we have

$$\begin{aligned} &\leq \sum_{i=1}^{n} |\xi_i|\,\|a_i\| \\ &\leq \left(\sum_{i=1}^{n} \|a_i\|\right) \max_{1\leq i\leq n} |\xi_i| \\ &= \left(\sum_{i=1}^{n} \|a_i\|\right) \|\xi\|_\infty\,, \\ \|T\xi\| &= \left\|\sum_{i=1}^{n} \xi_i a_i\right\| \end{aligned}$$

so T is bounded, hence continuous.

On the other hand, the unit sphere in $(\mathbb{R}^n, \|\cdot\|_\infty)$, namely

$$S = \{x \in R^n: ||x||_\infty = 1\},$$

is clearly a closed and bounded subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$, hence compact. Moreover, the function $\xi \mapsto ||T\xi||$ is continuous on R^n (why?) and hence on S; so it attains its minimum on S (why?) and this minimum value m cannot be zero (why?). So we have $||T\xi|| \geq m > 0$ for all $\xi \in S$, hence $||T\xi|| \geq m||\xi||_\infty$ for all $\xi \in R^n$ (why?).

It then follows from Corollary that T is a topological isomorphism.

Corollary 4.2 -On a finite-dimensional vector space, all norms are equivalent.

Proof. Let X be a vector space of finite dimension n, and let $||\cdot||_1$ and $||\cdot||_2$ be norms on X. Fix a basis $\{a_1,\ldots,a_n\}$ of X, and define the linear bijection $T: R^n \to X$ by (3.54).

By Proposition , T is a topological isomorphism $(\mathbb{R}^n, \|\cdot\|_\infty) \to (X, \|\cdot\|_1)$ and also $(\mathbb{R}^n, \|\cdot\|_\infty) \to (X, \|\cdot\|_2)$. But then id $= T \circ T^{-1}$ is a topological isomorphism from $(X, ||\cdot||_1)$ to $(X, ||\cdot||_2)$.

Corollary 4.3 - Every finite-dimensional normed linear space is complete.

Corollary 4.4 - Every finite-dimensional subspace of a normed linear space is closed.

Proposition 4.5 - Let X and Y be normed linear spaces, with X finite-dimensional. (Here the space Y is arbitrary.) Then every linear mapping of X into Y is continuous.

Proof. If X is an n-dimensional real normed linear space, then X is topologically isomorphic to $(\mathbb{R}^n, \|\cdot\|_\infty)$ by Proposition . On the other hand, any linear map from R^n to Y is of the form $T(\xi_1, \ldots, \xi_n) = \sum_{i=1}^n \xi_i y_i$ for suitable vectors $y_1,\ldots,y_n$ in Y (why?); and such a map is necessarily bounded [from $(\mathbb{R}^n, \|\cdot\|_\infty)$ to $(Y, \|\cdot|_Y)$] by the same calculation that was done during the proof of Proposition 3.21.

Theorem 4.6 - (F. Riesz's theorem) For a normed linear space X, the following are equivalent:

(a) The closed unit ball in X is compact.

(b) Every closed bounded set in X is compact.

(c) X is locally compact.

(d) X is finite-dimensional.

The proof of this theorem will rely on a simple but important lemma, also due to F. Riesz:

Lemma 4.7 - (F. Riesz's lemma) Let X be a normed linear space, and let M be a proper closed linear subspace of X. Then for each $\varrho > 0$ there exists a point $x \in X$ such that $||x||= 1$ and $d(x,M) \geq 1 - \varrho$.

Proof of F. Riesz's lemma. Since M is proper, there exists a point $x_1 \in X \setminus M$; and since M is closed, we have $d(x_1,M) = d > 0$. But this means that there exists $x_0 \in M$ such that $||x_1 - x_0||$ is arbitrarily close to d, say $\leq d/(1-\varrho)$. Now set

$$x = \frac{x_1 - x_0}{\|x_1 - x_0\|}.$$

By construction $||x|| = 1$. Moreover, for any $y \in M$ we have

$$\begin{aligned} \|x - y\| &= \left\| \frac{x_1 - x_0}{\|x_1 - x_0\|} - y \right\| \\ &= \frac{\|x_1 - (x_0 + \|x_1 - x_0\| y)\|}{\|x_1 - x_0\|} \end{aligned}$$

(why?). Since $x_0 + ||x_1 - x_0|| y \in M$, the numerator is $\geq d$ (why?); and the denominator is $\leq d/(1 - \epsilon)$; so the ratio is $\geq 1 - \epsilon$.

5.1. Introduction to duality for normed linear spaces

Among the spaces B(X,Y) of continuous linear mappings, there are two choices of the target space Y that are of particular importance:

- Y = R (or C in the case of a complex normed linear space), considered as a onedimensional vector space. This is obviously the simplest case.
- Y = X. In this case maps can be freely composed: therefore B(X,X), in addition to being a vector space, also has the structure of an operator algebra.

The case Y = X leads to the subject of spectral theory, which is of especial importance in quantum mechanics (among other applications) and which you are invited to study in a fourth-year course. Here I would like to begin the analysis of the case Y = R (or C), which plays a central role in the theory of Banach spaces; we will continue this study a few weeks from now, after proving the Hahn–Banach theorem. In what follows I will assume (simply to fix the notation) that the field of scalars is R, but everything here works with minor modifications when the field of scalars is C. (When we discuss the Hahn–Banach theorem we will have to treat the complex case separately.)

A linear mapping from a vector space X to the field of scalars (here R) is called a linear functional on X. The space L(X,R) of all linear functionals (continuous or not) on X is called the algebraic dual space of X, and is denoted $X^\sharp$. I assume that in your course in linear algebra you studied (even if only briefly) the theory of algebraic duals of finite-dimensional vector spaces, and proved in particular that the algebraic dual of an n-dimensional vector space is n-dimensional.[7] The theory of algebraic duals of infinite dimensional vector spaces belongs to a more advanced course in linear algebra.

In analysis, however, we are principally interested in continuous linear functionals. The space B(X,R) of all continuous linear functionals on a normed linear space X is called the (topological) dual space of X, and is denoted X^*. It is itself a normed linear space under the norm

$$\|\ell\|_{X^*} = \|\ell\|_{X\to\mathbb{R}} = \sup_{\|x\|_X\le 1} |\ell(x)|.$$

By , the normed linear space X^* is complete, i.e. is a Banach space; this holds whether or not X is complete. Moreover, Proposition if A is a dense linear subspace of X, then A^* is isometrically isomorphic to X^* under the obvious mapping.

Let us begin with some purely algebraic considerations. If X is a vector space (not necessarily normed) and M is a linear subspace of X (not necessarily closed), we say that M has codimension n in X (where n is a positive integer) if the quotient space X/M has dimension n, or in other words if there exist $e_1,\ldots,e_n \in X\setminus M$ such that every vector $x \in X$ can be uniquely represented in the form $x = \lambda_1 e_1 + \ldots + \lambda_n e_n + z$ where $\lambda_1,\ldots,\lambda_n$ are scalars and

$z \in M$. A linear subspace of codimension 1 is called a **(linear) hyperplane**. Note that in this case one can choose any vector in $X \setminus M$ to play the role of e_1 (why?).

If M is a (linear) hyperplane in X, then each coset $x_0 + M$ (where $x_0 \in X$) is called an **affine hyperplane** in X.[8]

The relevance of hyperplanes to linear functionals is explained by the following proposition, which is purely algebraic:

Proposition 5.1 - Let X be a vector space.

(a) The kernel of a nonzero linear functional $\ell \in X^\sharp$,

$$\ker\ell = \{x \in X : \ell(x) = 0\},$$

is a hyperplane in X. Conversely, every hyperplane $H \subset X$ is the kernel of some nonzero linear functional $\ell \in X^\sharp$.

(b) The set where a nonzero linear functional $\ell \in X^\sharp$ takes the value 1,

$$A_\ell = \ell^{-1}[1] = \{x \in X : \ell(x) = 1\},$$

is an affine hyperplane in X, not passing through the origin. Conversely, every affine hyperplane $A \subset X$ not passing through the origin is of the form A_ℓ for some nonzero linear functional $\ell \in X^\sharp$.

Proof. (a) Let ℓ be a nonzero linear functional on X. Then $\ker\ell$ is a proper linear subspace of X (why?). Now fix any $x_0 \in X \setminus \ker\ell$, and note that $\ell(x_0)$ 6= 0. We can then represent each $x \in X$ uniquely in the form $x = \lambda x_0 + y$ with $\lambda \in R$ and $y \in \ker\ell$; indeed, such a representation can hold only if we take $\lambda = \ell(x)/\ell(x_0)$ [just apply ℓ to both sides to see this], and if we do choose this value of λ, then $x - \lambda x_0 \in \ker\ell$ (why?). This shows that $\ker\ell$ is a hyperplane in X.

Conversely, if H is a hyperplane in X, choose any $x_0 \in X \setminus H$; then by hypothesis each $x \in X$ can be represented uniquely in the form $x = \lambda x_0 + y$ with $\lambda \in R$ and $y \in H$. Defining $\ell(x) = \lambda$ then does the trick.

(b) If ℓ is a nonzero linear functional on X, there clearly exists $x_0 \in X$ such that $\ell(x_0) = 1$ (why?). Then we have $A_\ell = x_0 + \ker\ell$ (why?), so using the result of part (a) we conclude that A_ℓ is an affine hyperplane in X; and clearly 0 ∈/ A_ℓ.

Conversely, if $A \subset X$ is an affine hyperplane not passing through the origin, then we can write $A = x_0 + H$ for some hyperplane H and some $x_0 \in X \setminus H$. By the result of part (a), there exists a nonzero linear functional $\ell_0 \in X^\sharp$ such that $H = \ker\ell_0$; in particular, $\ell_0(x_0)$ = 06 . Defining $\ell(x) = \ell_0(x)/\ell_0(x_0)$ then does the trick.

Let us now pass from algebra to analysis, and assume that X is a normed linear space.

Proposition 5.2 - In a normed linear space X, a hyperplane $M \subset X$ is either closed or dense.

Proof. By Proposition , $\overline{M}$ is a linear subspace of X containing M. Since M has

codimension 1, the only possibilities are $\overline{M} = M$ and $\overline{M} = X$.

Note that the affine hyperplanes $x_0 + M$ are also closed or dense according as M is.

Proposition 5.3 - Let X be a normed linear space, and let $\ell \in X^\#$ be a linear functional. Then ℓ is continuous if and only if $\ker\ell$ is closed. Moreover, we have the quantitative relation

$$\|\ell\|_{X^*} = \frac{1}{d(0, A_\ell)}$$

where

$$A_\ell = \ell^{-1}[1] = \{x \in X: \ell(x) = 1\}.$$

[If $\ell = 0$ we have $A_\ell = \emptyset$ and we define $d(0,\emptyset) = +\infty$.]

Proof. Everything is trivial (including the quantitative relation) if $\ell = 0$, so we henceforth assume that ℓ 6= 0.

If ℓ is continuous, then $\ker\ell$ is obviously closed. To prove the converse, suppose that $H = \ker\ell$ is a closed, and choose any $x_0 \in X \setminus H$ such that $\ell(x_0) = 1$; we then have $d(x_0,H) > 0$ (why?). Then every $x \in X$ can be uniquely written in the form $x = \lambda x_0 + z$ with $z \in H$ (why?), and such an x satisfies

$$\|x\| = d(x,0) \geq d(x,H) = |\lambda|\, d(x_0,H)$$

(why?) and

$$\ell(x) = \lambda$$

(why?), hence

$$|\ell(x)| \leq \frac{1}{d(x_0, H)} \|x\|,$$

which proves that ℓ is continuous.

The quantitative relation clearly holds when ℓ is discontinuous, because we have $\|\ell\|_{X\to R} = +\infty$ and $d_X(0,A_\ell) = 0$ (why?). When ℓ is continuous, we have

$$\|\ell\|_{X^*} = \sup_{x\neq 0} \frac{|\ell(x)|}{\|x\|} = \frac{1}{\inf\limits_{x:\,\ell(x)\neq 0} \frac{\|x\|}{|\ell(x)|}} = \frac{1}{\inf\limits_{x:\,\ell(x)=1} \|x\|} = \frac{1}{d(0, A_\ell)}$$

(you should make sure you understand each equality here).

In order to make further progress in the theory of dual spaces, we need to ensure that the dual space X^* is "large enough", i.e. that there are "enough" continuous linear functionals. What is "enough"? Well, first of all, we would like the continuous linear functionals to separate points of X: that is, for each pair $x,y \in X$ with $x = 6$ y, there should exist $\ell \in X^*$ such that $\ell(x)$ 6= $\ell(y)$. By linearity this is equivalent to the assertion that for every $x \in X$ with x 6= 0, there exists $\ell \in X^*$ such that $\ell(x)$ 6= 0. A few weeks from now we will deduce, as a corollary of the Hahn–Banach theorem, that this is indeed true, and indeed that ℓ can be chosen to make $\ell(x)$ as large as it is allowed to be:

Proposition 5.4 - (Corollary of Hahn–Banach theorem, to be proven later) Let X be a normed linear space. Then for each nonzero $x_0 \in X$, there exists a nonzero $\ell_0 \in X^*$ such that $\ell_0(x_0) = ||\ell_0||_{X^*} ||x_0||_X$.

Equivalently, we can choose $\ell_0 \in X^*$ with $||\ell_0||_{X^*} = 1$ such that $\ell_0(x_0) = ||x_0||_X$.

Let us go a little bit farther to explore one of the consequences of Proposition 3.31. Given a normed linear space, we can form the dual space X^* and then the second dual space $(X^*)^*$ [usually written X^{**} for short]. It is automatically complete, hence a Banach space.

Definition 5.5 - A normed linear space X is said to be reflexive if the natural embedding maps X onto X^{**}, i.e. if $\iota[X] = X^{**}$.

Definition 5.6 - Let X and Y be normed linear spaces, and let $T: X \to Y$ be a continuous linear mapping. Then the adjoint (or dual or transpose) map $T^*: Y^* \to X^*$ [note the reversal of order here!] is defined by

$$(T^*(\ell))(x) = \ell(Tx)$$

for $\ell \in Y^*$ and $x \in X$.

First of all, we should check that this definition makes sense! It is obvious that $(T^*(\ell))(x) = \ell(Tx)$ is linear in x, so $T^*(\ell)$ is indeed a linear functional on X. Moreover, we have

$$|(T^*(\ell))(x)| = |\ell(Tx)| \le ||\ell||_{Y^*} ||Tx||_Y \le ||\ell||_{Y^*} ||T||_{X\to Y} ||x||_X ,$$

so $T^*(\ell)$ is a bounded linear functional on X, of norm at most $k\ell k_{Y^*} kTk_{X\to Y}$. Finally, the map $T^*: Y^* \to X^*$ given by $T^*: \ell \to 7\; T^*(\ell)$ is manifestly linear; and we have just shown that it is a bounded linear operator, whose operator norm satisfies

$$||T^*||_{Y^*\to X^*} \le ||T||_{X\to Y}$$

(why?). So we indeed have $T^* \in B(Y^*, X^*)$, as desired.

Lemma 5.7 - We have $||T^*||_{Y^*\to X^*} = ||T||_{X\to Y}$.

Proof. We have just shown that $||T^*||_{Y^*\to X^*} \le ||T||_{X\to Y}$, so we need only show the reverse inequality. So let $x \in X$. Then by Proposition 3.31 there exists $\ell \in Y^*$ such that $||\ell||_{Y^*} = 1$ and $\ell(Tx) = ||Tx||_Y$. Therefore $||Tx||_Y = \ell(Tx) = (T^*(\ell))(x) \le ||T^*(\ell)||_{X^*} ||x||_X$

$\leq ||T^*||_{Y^* \to X^*} ||\ell||_{Y^*} ||x||_X = ||T^*||_{Y^* \to X^*} ||x||_X$.

Since this holds for every $x \in X$, we have shown that $||T||_{X \to Y} \leq ||T^*||_{Y^* \to X^*}$.

Lemma 5.8 -

(a) If $T: X \to Y$ is a topological isomorphism, then $T^*: Y^* \to X^*$ is also a topological isomorphism, and in fact $(T^*)^{-1} = (T^{-1})^*$.

(b) If $T: X \to Y$ is an isometric isomorphism, then $T^*: Y^* \to X^*$ is also an isometric isomorphism.

Proof. (a) Let T be a topological isomorphism, and let us write $S = T^{-1}$. Since by hypothesis $S \in B(Y,X)$, the adjoint $S^* \in B(X^*,Y^*)$ is well-defined. Then, for any $x \in X$ and $\ell \in X^*$, we have by definition

$$(T^*(S^*(\ell)))(x) = (S^*(\ell))(Tx) = \ell(S(Tx)) = \ell(x)$$

(why?), so that $T^*(S^*(\ell)) = \ell$, i.e. $T^* \circ S^* = I_{X^*}$. A similar calculation shows that $S^* \circ T^* = I_{Y^*}$. Hence T^* and S^* are inverses, as was to be shown.

(b) Let T be an isometric isomorphism. Then we have just shown that T^* is a topological isomorphism; we need only show that it is an isometry. But this is easy: since T is an isometric isomorphism, we have $||T||_{X \to Y} = 1$ and $||T^{-1}||_{Y \to X} = 1$. But then by Lemma 3.35 we have $||T^*||_{Y^* \to X^*} = ||T||_{X \to Y}$ and $||(T^*)^{-1}||_{X^* \to Y^*} = ||(T^{-1})^*||_{X^* \to Y^*} = ||T^{-1}||_{Y \to X} = 1$ (why does the first equality hold?). So T^* is an isometry.

Much more can be said about dual spaces and adjoint operators, but this is enough for now.

5.2. Duals of the sequence spaces c_0 and ℓ^p $(1 \leq p < \infty)$

We are now ready to determine the duals of the sequence spaces c_0 and ℓ^p for $1 \leq p < \infty$ (but not ℓ^∞, which is more difficult). We shall do this "with our bare hands", i.e. we shall not use the not-yet-proven Proposition or any other "deep" result.

The intuition behind all of this is the following elementary fact: every linear functional on R^n is of the form

$$\iota(a): x \mapsto \sum_{i=1}^{n} a_i x_i$$

for some unique $a \in R^n$; and conversely, every $a \in R^n$ defines a linear functional $\iota(a)$ on R^n by the above formula.

It is natural to try to adapt this idea to spaces of infinite sequences, defining linear functionals by the formula but with the sum now taken to infinity. But now we have to be careful about the precise spaces in which x and a are taken to lie, and to make sure that the infinite sum is convergent and defines a bounded linear functional.

Let us start with the case X = c_0 (equipped as usual with the sup norm). We will show that the dual X* can be naturally identified with the space ℓ^1. Here and in what follows, we will write e_i to denote the infinite sequence having a 1 in the ith coordinate and 0 elsewhere.

Proposition 5.9 - For any a ∈ ℓ^1, the sum

$$\sum_{i=1}^{\infty} a_i x_i$$

is absolutely convergent for all x ∈ c_0, and defines a linear functional

$$\iota(a): x \mapsto \sum_{i=1}^{\infty} a_i x_i$$

on c_0 that is bounded (i.e. belongs to c_0^*) and has nnor

$$\|\iota(a)\|_{c_0^*} = \|a\|_{\ell^1}.$$

Moreover, every bounded linear functional on c_0 arises in this way, and the map ι is an isometric isomorphism of ℓ^1 onto c_0^*.

Proof. Fix a ∈ ℓ^1. Then for all x ∈ c_0 we have

$$\sum_{i=1}^{\infty} |a_i x_i| \le \|a\|_{\ell^1} \|x\|_{c_0}$$

(why?), so the sum $\sum^{\infty} a_i x_i$ is absolutely convergent and satisfies

$$\left|\sum_{i=1} a_i x_i\right| \le \sum_{i=1} |a_i x_i| \le \|a\|_{\ell^1} \|x\|_{c_0}.$$

It follows $\iota(a) \in c_0^*$ and that $\|\iota(a)\|_{c_0^*} \le \|a\|_{\ell^1}$ that.

To see the reverse inequality, consider for each n the vector $x^{(n)} \in c_{00} \subset c_0$ defined by

$$x_i^{(n)} = \begin{cases} \operatorname{sgn}(a_i) & \text{for } 1 \le i \le n \\ 0 & \text{for } i > n \end{cases}$$

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where

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

We then have $||x^{(n)}||_{c0} \leq 1$ (why?) and

Where.

$$\iota(a)(x^{(n)}) = \sum_{i=1}^{n} a_i \,\mathrm{sgn}(a_i) = \sum_{i=1}^{n} |a_i|$$

The map $\iota : \ell^1 \to c_0^*$ is clearly linear and injective (why is it injective?), so it remains only to show that it is surjective, i.e. that every $\ell \in c_0^*$ is of the form $\iota(a)$ for some (necessarily unique) $a \in \ell^1$. Indeed, we claim that $\ell = \iota(a)$ where a is given by the formula

$$a_i = \ell(e_i)$$

(recall that $e_i \in c_0$ is the vector having a 1 in the ith coordinate and 0 elsewhere). To see this, note first that if $x = (x_1, x_2, \ldots) \in c_0$, then

$$\left\| x - \sum_{i=1}^{n} x_i e_i \right\|_{c0} = \sup_{i>n} |x_i| \to 0$$

as $n \to \infty$ (why?), i.e. $\sum_{i=1}^{n} x_i e_i \to x$ in c_0 as $n \to \infty$. Now consider any $\ell \in c_0^*$; since ℓ is continuous, we have

iP=1

$$\ell\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \,\ell(e_i) \to \ell(x)$$

as $n \to \infty$, or in other words

$$\ell(x) = \sum_{i=1}^{\infty} a_i x_i$$

where the sum is convergent (but we have not yet proven that the convergence is absolute). The final step is to prove that $a \in \ell^1$ (this will imply in particular that the convergence is absolute). To see this, let vectors $x^{(n)} \in c_{00} \subset c_0$ be defined once again and recall that $||x^{(n)}||_{c0} \leq 1$. Then

$$\sum_{i=1}^{n} |a_i| = \sum_{i=1}^{n} a_i \,\mathrm{sgn}(a_i) = \ell(x^{(n)}) \leq \|\ell\|_{c_0^*}$$

so taking n → ∞ we conclude that a ∈ ℓ^1 (with $\|a\|_{\ell^1} \leq \|\ell\|_{c_0^*}$). It now follows that $\ell = \iota(a)$ (and hence, from the first part of the proof, that $\|a\|_{\ell^1} = \|\ell\|_{c_0^*}$).

Corollary 5.10- The spaces ℓ^p for $1 < p < \infty$ are reflexive.

Lemma 5.11 - $\iota_q^* \circ J_p = \iota_p$.

To start with, you should check that the statement of this lemma makes sense, i.e. that the maps act between the correct spaces. We can then prove the lemma as follows:

Proof. We need to show that $(\iota^*_q \circ J_p)(x) = \iota_p(x)$ for all x ∈ ℓ^p, or in other words that

$$((\iota_q^* \circ J_p)(x))(y) = (\iota_p(x))(y)$$

for all x ∈ ℓ^p and all y ∈ ℓ^q. Now

$$(\iota_p(x))(y) = \sum_{i=1}^{\infty} x_i y_i$$

by definition of ι_p. On the other hand, we have

$$((\iota_q^* \circ J_p)(x))(y) = (\iota_q^*(J_p(x)))(y) = (J_p(x))(\iota_q(y)) = (\iota_q(y))(x) = \sum_{i=1}^{\infty} y_i x_i \qquad (3.103)$$

by the definitions of adjoint, J_p and ι_q.

Proof of Corollary 3.39. By the lemma we have $\iota_q^* \circ J_p = \iota_p$. But by Lemma 3.36(b), the map ι_q^* is an isometric isomorphism. Then $J_p = (\iota_q^*)^{-1} \circ \iota_p$ is an isometric isomorphism, as was to be proved.

6.1. Fuzzy strong ϕ-b-normed linear space

In this section we give the definition of fuzzy normed linear space in a new approach.

Definition 1 . Let ϕ be a function defined on R to R with the following properties

(ϕ1) $\phi(-t) = \phi(t)$, $\forall t \in R$;

(ϕ2) $\phi(1) = 1$;

(ϕ3) ϕ is strictly increasing and continuous on $(0,\infty)$;

Definition 2. Let X be a linear space over a field F and $K \geq 1$ be a given real number. A fuzzy subset N of $X \times R$ is called fuzzy strong ϕ-b-norm on X if $\forall x,y \in X$ the following conditions hold:

(bN1) $\forall t \in R$ with $t \leq 0$, $N(x,t) = 0$;

(bN2) ($\forall t \in R$, $t > 0$, $N(x, t) = 1$) iff $x = \theta$;

(bN3) $\forall s, t \in R$, $N(x + y, s + K t) \geq N(x, s) * N(y,t)$;

(bN4) $N(x,\cdot)$ is a non-decreasing function of t and

$\lim_{t\to\infty} N(x,t) = 1.$

Then $(X, N, \phi, K,*)$ is called fuzzy strong ϕ-b-normed linear space.

Example 3. Consider the linear space R and a fuzzy subset N of $R \times R$ by

$$N(x,t) = \begin{cases} \exp(-\frac{|x|^p}{t}) & t > 0 \\ 0 & t \leq 0 \end{cases}$$

for all $x \in R$ and $0 < p \leq 1$ and consider the t-norm $*$ by $a * b = ab$, $\forall a,b \in R$. Now,

(i) Clearly (bN1) holds from the definition.

(ii) $(\forall t \in \mathbb{R}, t > 0,\ N(x,t) = 1) \iff \exp(-\frac{|x|^p}{t}) = 1 \iff |x|^p = 0 \iff x = 0$

Therefore (bN2) holds.

(iii) Clearly $N(x,\cdot)$ is a non-decreasing function of t and

$\lim_{t\to\infty} N(x,t) = 1.$

Hence $(X,N,\phi,K,*)$ is a fuzzy strong ϕ-b-normed linear space where $K = 2^p (> 1)$ and $\phi(\alpha) = |\alpha|^p, \forall \alpha \in R$, $0 < p \leq 1$.

7.1. Finite Dimensional Normed Linear Spaces

Note. In this section, we classify a normed linear space as finite dimensional by considering the topological property of compactness.

Note. Each of the following is a norm on R^2 (here we denote $x \in R^2$ as (x_1,x_2)):

(i) the sup norm: $||x||_\infty = ||x||sup = \max\{|x_1|,|x_2|\}$

(ii) the Euclidean norm: $||x|| = {}^{P}(x_1)^2 + (x_2)^2$

(iii) the absolute value deviation norm (also called the "taxicab norm"): $||x||_1 = |x_1| + |x_2|$.

Notice the appearance of unit balls under these norms in Figure 2.3. In fact, we will show that all norms on a finite dimensional space are equivalent (in Theorem 2.

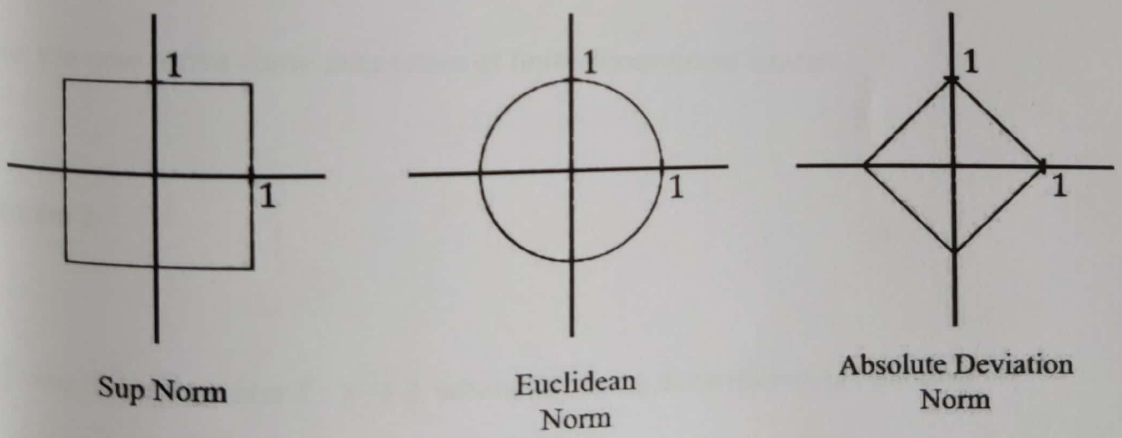

Figure . Unit balls under various norms.

Note. In this section, $F = \{1,2,\ldots,N\}$ and $B(F)$ is the set of bounded functions on F:

$$f \in B(F) \Rightarrow f : F \to F \text{ and } \{f(s) \mid s \in F\} \text{ is bounded.}$$

is a basis of B(F) since any $f \in B(F)$ can be uniquely written as

$$f(x) = \sum_{i=1}^{N} f(i)\delta_i(x)$$

Proposition 1. For any normed linear space Z, all elements of L(B(F),Z) (the set of linear operators from B(F) to Z) are bounded.

Note. The familiar Heine-Borel Theorem states that a set of real numbers is compact if and only if it is closed and bounded. We will see this violated in infinite dimensional spaces. However, "half" of Heine-Borel holds in B(F). The other half holds by Notice that F is a finite set here.

Theorem 1 Closed and bounded subsets of B(F) are compact.

Proposition 2 If T is a bijective linear operator from the normed linear space X to B(F), then T is bounded.

Note. We now prove some properties of finite dimensional spaces.

Theorem 2

(a) Any linear operator $T : X \to Z$, where X is finite dimensional, is bounded.

(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field F (where F is R or C) are complete.

(c) Any finite-dimensional subspace of a normed linear space is closed.

(d) **Theorem 3.** A linear operator $T : X \to Y$, where Y is finite-dimensional, is bounded if and only if N(T) (the nullspace of T) is closed.

Note. We want to talk about "orthogonality" (or "perpendiculars"). However, in a normed linear space, we do not necessarily have an inner product ("dot product") and so we cannot measure angles. However, inspired by our experience in Euclidean space R^n (and the usual Euclidean norm), we have the following definition concerning perpendiculars to subspaces.

Definition. Given a proper subspace M of the normed linear space $(X, ||\cdot||)$, a nonzero vector $x \in X$ is perpendicular to M if

$$d(x,M) = \inf\{||x - y|| \mid y \in M\} = ||x||.$$

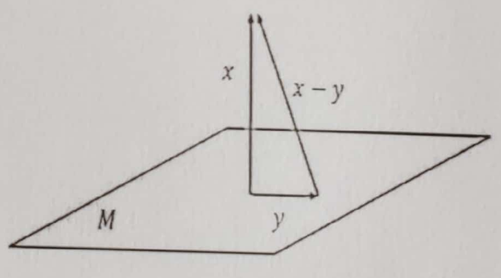

Figure. The idea of "perpendicular" defined using norms.

Theorem 4. Riesz's Lemma.

Given a closed, proper subspace M of a normed linear space $(X, ||\cdot||)$ and given $\varepsilon > 0$, there is a unit vector $x \in X$ such that $d(x,M) \geq 1 - \varepsilon$.

Note. The following result is particularly intriguing! It relates the algebraic idea of dimension to the topological idea of compactness. It also hints at the fact that the Heine-Borel Theorem only holds in finite dimensional spaces.

Theorem 5. Riesz's Theorem. A normed linear space $(X, ||\cdot||)$ is finite-dimensional if and only if the closed unit ball $B(0;1)$ is compact.

8. Conclusion

In a nutshell, we have introduced the product-normed linear space and product-semi-normed linear space (product-semi-Banch space) which are endowed with some functional space. In addition, P-NLS is endowed with fixed point.

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Polya's Counting Theory

A Project submitted

To the

University of Mumbai

for the

M.Sc. Degree

in Mathematics

SUBMITTED BY

Mishra Rahul Kumar Mithilesh

UNDER THE GUIDANCE OF

Prof. DIPALI MALI [HOD] OF MATHEMATICS

Prof. HUSNAWAZ CONTRACTOR

Department of Mathematics ,

(University of Mumbai) ,

Sonopant Dandekar College ,

Palghar – 401404.

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MAY 2023

STATEMENT TO BE INCORPORTED BY THE CANDIDATES IN THE PROJECT AS REQUIRED UNDER REGULATION FOR THE M.Sc. DEGREE

As required by the university Regulation No. R.1972/R.2190 we wish to state that the work embodied in this project titled "Polya's Counting Theory" is expository in nature and doesn't contain any new results . The entire work is carried out under the guidance of Prof. Dipali Mali & Prof. Husnawaz Contractor at the Department of mathematics , SDSM college . This work has not been submitted for any other degree of this or any other university . Whenever references have been made of previous work of others , it has been clearly indicated as such and included in the Bibliography .

Place: SDSM College , Palghar .

Date : May 2023

Signature of Student

Mishra Rahul Kumar Mithilesh

Certificate

This is to certify that Mishra Rahul Kumar Mithilesh has worked under my supervision for a Master of Science project titled "Polya's Counting Theory" . The project is an expository in nature and does not contain any new results. Also the project has not been submitted to this University or any other University for any other degree.

Place: SDSM College , Palghar.

Date : May 2023

26/05/23 26/05/23 (DIPALI MALI)

Signature of Guide

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I also thank all our friends and family members for supporting me throught this endeavour.

Place: SDSM College , Palghar.

Date : May 2023

SDSM College , Palghar – 401404,
University of Mumbai, Mumbai – 400 098.

Signature of Candiates:

Mishra Rahul Kumar Mithilesh

Declaration

We declare that this written submission represents our ideas in our own words and where other's ideahave been included, We have adequately cited and referenced the original sources. We also declare that we have adhered to all principles of Academic honesty and integrity and have not misrepresented or fabricated or falsified any of the ideas / data / fact / source in our submission . We understand that any violation of the above will cause disciplinary action by the Institute and can also evoke penl action from the sources which thus not been properly cited or from whom proper permission has not been taken when needed.

RKishra

Signature

Mishra Rahul Kumar Mithilesh

Date: May 2023

Abstract:

The Pólya enumeration theorem, also known as the Redfield–Pólya theorem and Pólya counting, is a theorem in combinatorics that both follows from and ultimately generalizes Burnside's lemma on the number of orbits of a group action on a set. The theorem was first published by J. Howard Redfield in 1927.

CONTENT

Polya's Counting Theory

1 Introduction

Polya theory is, unlike most of high- school combinatorics, not a bag of tricks that are situation- specific. It deals with questions where clearly understanding the set that is to be counted is the main difficulty.
An example of this is when we are trying to find the number of nationalities represented in a group of people: having counted the first Indian, we must ignore the other Indians as the counting proceeds.
In this case certain groups of people were related by nationality; however, the problems of Polya theory involve relations of symmetry operating among elements (people), and the crux of their solution is to understand how symmetry changes the relationships between these elements.
In what follows we will consider symmetry problems from elementary combinatorics, such as necklaces and chessboards, as well as solid geometry and chemistry, and we will hint at the number theoretic applications that abound in this field. The full scope of Polya theory is, of course, far greater, and is visible in graph theory and higher algebra as well.

Remark 1.1 We begin with a simplified chessboard example that, although easily solved by brute force, quickly generalises to problems that beg for more systematic methods of solution.

Example 1.2 Consider a 2×2 board whose squares are coloured either in red(r) or in black(b). We must find the number of different boards. Elementary counting suggests that since each square can have one of two colours, there are $2^4 = 16$ possible arrangements. A lot, however, depends on what we mean by 'different'. The consensus among most people is that two boards are equivalent if one can be obtained from the other by a rotation of some multiple of 90 anticlockwise. Assuming this, we find the following arrangements: a) 4 red squares b) 3 red, 1 black c) 2 red, 2 black (there are 2 possiblities here, where similar squares are on the same half or the same diagonal).

Thus the total number of non-equivalent boards is $2\cdot(1+1)+2=6$, since the two remaining cases are (a) and (b) with the colours interchanged. The reader will appreciate the intrinsic difficulty of the problem immediately upon replacing the ' 2×2 ' even with ' 3×3 ', and certainly with any greater number. Problems of a similar nature abound in elementary settings: The number of necklaces of n beads, with a choice of m colours, where necklaces with rotational symmetry are equivalent; the number of colourings of

the vertices of a cube or tetrahedron in m colours, equivalent up to rotation; and several others that will be returned to.

2 Abstracting the problem

In each case mentioned above, there is an assignment of colours to squares, beads or vertices- a *function* from a domain to the set of colours. Elementary counting gives us the cardinality of the set of all such functions R^D, where D is the domain and R, the range. However, some functions are 'equivalent' (the elucidation of 'equivalence' being central to our study and done presently) and thus the real problem is to determine the number of *eqivalence classes* of functions on R^D.

So, when are two functions equivalent? Returning to our examples, we see that equivalence was defined by the property that one arrangement (function) could be obtained from another by an operation on the elements of the domain. What sort of operation? Clearly, any such operation *permutes* the elements among themselves, otherwise the power set is ill defined. Furthermore, as in our example, if we can speak of rotating a chessboard by multiples of 90, we must surely include rotations by zero (the identity operation) and negative multiples (inverses). We see that the set of operations possess a group structure; and, as our discussion will show, it is sufficiently general to consider that the operations form a permutation group.

We will formalize the concept of equivalence of functions. Let G be a permutation group acting on a set D. There exists a set R and the set R^D, the set of all functions from D to R.

Definition 2.1 *Two functions $f,g \in R^D$ are equivalent if there exists a permutation $\pi \in G$ such that for all $d \in D, f(d) = g(\pi(d))$. Also, for every $\pi \in G$, define $e_\pi : R^D \rightarrow R^D$ such that $e_\pi(f) = g$ iff f and g are equivalent.* e_π is well defined. For, if for some i, $f(\pi^{-1}(i) = g(i)$, then setting $i = \pi(d)$, which is possible as π is a permutation, we get $f(d) = g(\pi(d))$.

Further, $e(\pi)$ is injective. If $e_\pi(f) = e_\pi(h) = g$, then $f(d) = h(d)$ for all $d \in D$ and so $f = h$. This, together with the fact that the domain and range have equal cardinality, imply that e_π is also surjective and so is a permutation on R^D.

The equivalence class of f is called a *pattern*. Thus, the set of chessboards that form a pattern can be obtained from each other by rotation. Our objective is to enumerate the patterns on R^D.

3 A Partial Answer- Burnside's Lemma

The full machinery that we seek to employ is not always necessary, at least when we set ourselves the limited problem of counting the number of patterns without asking for any further information about their nature or their constituent functions.

Consider a group G acting on a set S. For every $g \in G$, let S^g denote the subset of S fixed by g. Also for every $s \in S$ define the *stabilizer* G_s of s, the subgroup of G (this is easily proved to be a subgroup) which fixes s, so that $gs = s$ for all $g \in G_s$. Now we find the cardinality of the set $(g,s) : gs = s$.

Fixing a g and then summing over G gives $\sum_{g \in G} |S^g|$. Alternatively, fixing an s and summing over S yields $\sum_{s \in S} |G_s|$. The latter sum can be simplified with the Orbit- Stabilizer Theorem. Indeed,

$$\begin{aligned}\sum_{s \in S} |G_s| &= \sum_{s \in S} \frac{|G|}{|O_s|} \\ &= |G| \sum_{s \in S} \frac{1}{|O_s|} \\ &= |G| \sum_{orbits O} \frac{|O_s|}{|O_s|} \\ &= |G| \cdot N,\end{aligned}$$

where N is the number of orbits of S and $|O_s|$ is the order of the orbit of s. This gives an expression for N:

Lemma 3.1 (Burnside's Lemma) *Given a group G acting on a set S, the number of orbits N due to this action is given by*

$$N = \frac{1}{|G|}\sum_{g\in G}|S^g|.$$

Remark 3.2 It is rather interesting to find that this is just as often often referred to as the *not- Burnside Lemma* in recognition of its prior discovery independently by Cauchy and Frobenius.

We immediately apply Burnside's Lemma to Example 1.2 letting G be the group C_4, the cyclic group of rotations by multiples of 90^0. We number the squares as quadrants. Letting S^θ denote the set of coloured boards (functions) which remain unchanged upon rotation by θ, we see that

a) Any board is fixed by rotation through $\theta = 0$. There are 16 such boards and so $|S^0 = 16$.

b) A 90^0 rotation sends 1 to 2, 2 to 3, 3 to 4, 4 to 1. If the board is unchanged, the colour of square 1 should be the same as that of 2, and similarly for all pairs. We thus have a monochromatic board, giving $|S^{90}| =$ 2(red or black).
Symmetry gives $|S^{270}| = 2$ as well.

c) A 180^0 rotation is a reflection about the x- axis and sends 1 to 4, 2 to 3. Arguing as before, we get $|S^{180}| = 2^2 = 4$.

Burnside's Lemma now gives $N = \frac{1}{4}(16 + 2 + 2 + 4) = 6$, which we had obtained previously.

However, there are some clear limitations to the use of this lemma: computing fixed points of sets under group action is often tedious and does not allow us to solve problems such as the following:

Problem: Find the number of different cubes that can be obtained by colouring its vertices in 2, 3, m colours. Two cubes are considered equivalent under rotational symmetry.

4 A Better Understanding of Equivalent Functions; The Cycle Index

We need a better understanding of the properties of a function that is fixed by a permutation in G. A key insight is the following lemma, which restricts the values of the functions that can be fixed by a permutation $\pi \in G$.

We will use the fact that under action of π,we obtain a permutation of the elements of the domain which can be decomposed into cycles. If there are b_i cycles of length i, the permutation is said to have *cycle type* $(b_1,b_2,...,b_n)$.

Consider one of the cycles of length j, $C = (a_1,a_2,..,a_j)$. We have the following lemma:

Lemma 4.1 *If π fixes a function f on R^D then f is constant for all $a_i \in C$.*

Proof: The above is equivalent to saying that $e_\pi(f) = f$ and so for all $d \in D, f(d) = f(\pi(d))$. But if d is in C, ie. $d = a_i$ for some i, then $f(a_i) = f(\pi(a_i)) = f(a_{i+1})$ and so f is constant within C.

In the language of example 1.2, if upon rotation by 90˚, which generates a single cycle of length 4 among the four squares, if we get an unchanged board then for all squares in the cycle (basically, all 4 squares), the value of the function (the colour assigned to the square) must be constant for all elements of the cycle (squares).

Upon rotation by 180˚ we get two cycles, (14) and (23). So within each cycle, any assignment must be constant over the cycle. How this is to be used to count the patterns will be deferred until we have a scheme to represent assignments suitably.

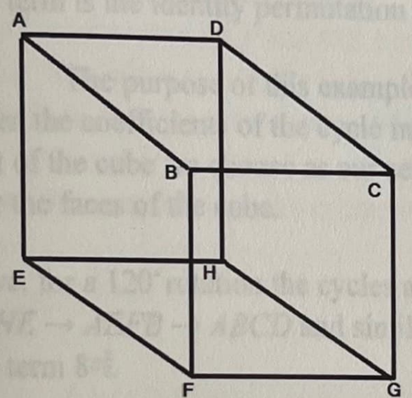
Moreover, in order to use our knowlegde of Lemma 4.1 we will need some sort of bookkeeping device that stores information about the cycles generated separately by all the elements of G. The 'device' most suitable for computation is a polynomial whose each term describes one element of G, and so we define the following:

Definition 4.2 (Cycle Index) For each $\alpha \in G$ with $|G| = n$ define a *monomial* $x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$ where the x_i are arbitrary variables and b_i is the number of cycles of length i in α. Then the *cycle index* of G in its action on set S is defined as

$$P_G(x_1, x_2, ..., x_n) = \frac{1}{|G|} \sum_{\alpha \in G} (monomial(\alpha))$$

Since calculating the cycle index of our groups is fundamental to what is to come, it is worthwhile to perform the calculation for some well- known objects.

Example 4.3 Now let G be the group of rotations (rotational symmetries) of a cube $ABCDEFGH$. The total number of symmetries, $|G|$, is 24. Indeed, a vertex can be rotated to any other face under the action of G, so its orbit has order 8. Also, its stabiliser is the group of rotations by multiples of 120^0 about the axis passing through the vertex and the centre of the cube, and has order 3. The Orbit-Stabilizer Theorem now gives $|G| = 8 * 3 = 24$. The rotations in G can be classified as follows:



a) **Rotations fixing vertices:** A rotation fixing, say, A and G can either send $E \rightarrow B \rightarrow D \rightarrow E$ (120° rotation) or can send $E \rightarrow D \rightarrow B \rightarrow E$ (240° rotation). We recognize that these two rotations, along with the identity, constitute the stabilizer of A and G, mentioned previously. Thus, for each such rotation, there are two 1-cycles and two 3-cycles, to which we assoociate the monomial $x_1^2x_3^2$. There are two rotations per pair of opposite vertices and 4 such pairs in the cube, giving a factor of 8 to the above monomial.

b) **Rotations fixing edges:** Suppose we rotate the cube by 180° about the axis formed by joining the midpoints of AB and GH. Then, we have the cycles (AB), (GH), (DF), (CE) and the associated monomial x_2^4. There are 6 axes, one for each pair of opposite edges.

c)Rotations fixing faces: Here the axis is through the centre of a face and perpendicular to it. For rotations by 90° or 270°, the vertices that bound these faces are permuted among themselves in two 4-cycles, ie. $2 \cdot x_4^2$. For a 180°-rotation, there are four 2- cycles formed by vertices on the same face, diagonal to each other. We can choose the axis in 3 ways. This gives us the cycle index of *G*:

$$P_G(x_1, x_2, .., x_n) = \frac{x_1^8 + 8x_1^2x_3^2 + 9x_2^4 + 6x_4^2}{24}$$

where the first term is the identity permutation monomial.

Example 4.4 The purpose of this example is to show that in addition to the degree, even the coefficients of the cycle index can change depending on which element of the cube we choose as our set. Let us perform the same calculation for the faces of the cube.

a) As above, for a 120° rotation the cycles are (A)(G)(EBD)(FCH). So *ABCD* → *ADHE* → *AEFB* → *ABCD* and similarly for the other three faces. This gives the term $8x_3^2$.

b) Using the cycle decomposition for vertices, we have *ABCD* → *BAEF*,*ADHE* → *BFGC*,*DCGH* → *FEHG* with the associated term $6x_2^3$.

c) For 90° and 270° rotations, two faces are fixed and the others go around in a 4-cycle. For 180°, the 4-cycle breaks into two 2-cycles. The contributionto the cycle index reads $6x_1^2x_4 + 3x_1^2x_2^2$.

Thus, $[P_G(x_1, x_2, ..., x_n) = \frac{x_1^6 + 8x_3^2 + 6x_2^3 + 6x_1^2x_4 + 3x_1^2x_2^2}{24}.]$

Example 4.5 We shall try to find the cycle index of the cyclic group C_n of rotations acting on a regular n- gon- a situation encountered in the necklace problem outlined along with Example 1.2. First we observe that this group is isomorphic to the additive group *Zn* of integers modulo *n*, acting on the set $[n] = (1,2,...,n)$.

Next, we see that the difference between consecutive numbers in a cycle is the same by definition, and for this difference *d*, the length of the cycle is the least number *k* such that $kd \equiv 0 \pmod{n}$. For a fixed *d*, this *k* is unique and so the length of all cycles is the same. Thus, if there is a cycle of length *k*, then there are *n/k* cycles, partitioning *Zn*. Of course $k|n$ by this argument.

Now, we must find the number of values of *d* that act on $[n]$ in this manner, for a fixed *k*. For the above congruence to hold clearly $d = \frac{n}{k} \cdot t$ for some $t \leq k$. Notice that with this value of *d*, the number $k' = \frac{k}{(t,k)}$ is also a solution to the congruence, because the product k^0d is still a multiple of *n*. Since *k* is the least solution for fixed *d* we must have $(t,k) = 1$ and since $t < k$ there are exactly $\varphi(k)$solutions, ie. $\varphi(k)$ permutations with cycle length *k*.

n/k

These contribute the term $\varphi(k)x_k$ to the cycle index. Thus, we finally have our expression:

$$P_G(x_1, x_2, .., x_n) = \frac{\sum_{k|n} \phi(k) x_k^{n/k}}{n}.$$

Corollary Since the sum of coefficients in the cycle index is by definition 1, we obtain $\text{P}_{k|n}\, \varphi(k) = n$.

5 What do we really want now? Weight for it . . .

To summarize what we have done so far, we have restated the problem of assigning properties to sets as a problem of finding the number of patterns, or classes of functions within a power set, some of which are equivalent under the action of a permutation group. The number of patterns is closely related to the number of functions that are fixed by the permutations of the group. Burnside's Lemma is the clearest example of this. Furthermore, if a permutation fixes a function we have proved it must be constant over its individual cycles. All the information we could possibly need about cycles is contained in the cycle index, which is a lot easier to compute than the fixed points in Burnside's Lemma. So we have the tools we need to find the

number of patterns in a group. But with the amount of machinery we have at our disposal, a little more effort can actually allow us to solve a much bigger question: What is the exact nature of the patterns? How many of the squares in Example 1.2 have equal numbers of red and black? How many cubes exist where exactly three vertices are coloured red, and the rest blue?
The new idea we present is motivated by the need to distinguish between different elements of the range R. We define for each $r \in R$ a *weight function* $w : R \to \mathbb{R}$. Also, we define the following:

Definition 5.1 The weight of a function $f : D \to R$ is given by

$$w(f) = \prod_{d \in D} f(d).$$

Also, the total weight of R is $w(R) = \sum_{r \in R} w(r)$.
For example, if there are six mathematicians to each of whom one problem from a selection of hard problems is assigned, if three are given the Goldbach Conjecture, two the P/NP problem and one the Riemann Hypothesis, the weight assignment (using an obvious convention) for this function would be g^3p^2r. Our definition of weights has the following very desirable property:

Proposition 5.2 *Equivalent functions have equal weight.* Proof: Suppose $e_\pi(f) = g$. Then for all $d \in D, f(d) = g(\pi(d))$ and thus

$$w(f) = \prod_{d \in D} f(d)$$

$$= \prod_{d \in D} f(\pi^{-1} d)$$

$$= \prod_{d \in D} g(d)$$

$$= w(g)$$

This allows us to make the following definitions:

Definitions 5.3 For each pattern F, define $w(F) = w(f)$ for some $f \in F$. By Proposition 5.2, this function is well-defined.
Now define the *pattern inventory*, which is given by

$$P.I. = \sum_{F} w(F)$$

where the sum covers all patterns in R^D.
At last we have clearly defined what we were looking for: The pattern inventory contains by means of its weighted terms all the information about the action of the group that we could possibly need. The number of patterns, which we calculate when we have considered all patterns to be equally important, is indeed obtained by setting all weights equal to unity. Knowledge of the pattern inventory is, in principle at least, the solution to all the problems we have set ourselves hitherto.
To see this, we consider the following example:

Example 5.4 We are to place marbles into a container with three holes arranged in the form of an equilateral triangle. There are 6 marbles in total. We must list all possible arrangements, assuming that the dihedral group D_3 acts on the triangle.
Here, if we assign weight a_i to a group of i marbles in the same hole, the possible arrangements are given in the pattern inventory

$$P.I. = a_6 + a_5a_1 + a_4a_1^2 + a_4a_2 + a_3^2 + a_3a_2a_1 + a_2^3.$$

We leave it to the reader to verify that the pattern inventory for Example 1.2, assuming that red and black are weighted with r and b respectively, is $P.I. = r^4 + r^3b + 2r^2b^2 + rb^3 + b^4$.
This example reminds us of something very important:

Remark *Different patterns may have the same weight.*

We now have everything we need to find the pattern inventory purely by examining group actions on the given domain. There is a final link, the relation between a function constant over given subsets of the domain and its weight, which is given by the following lemma:

Lemma 5.5 For a function $f : D \rightarrow R$ let $D = {}^{S}{}_{i} D_i$ such that f is constant over any given D_i. Then the total weight of all such functions f is

$$W = \text{Y}(\text{X}\ w(r)|Di|)\ i\ r \in R$$

Proof: Any such function has the weight $w_1^{|D_1|} w_2^{|D_2|} \dots w_i^{|D_i|} \dots$ and so belongs to the expression W. Conversely, any term in W corresponds to the unique function f with $f(d) = w_i$ for all $d \in D_i$.

Thus, if our six mathematicians (see def. 5.1) asked for a reallocation of problems but demanded that their teammates remain unchanged, even if two teams got the same problem, the possible weights of functions in this case would be terms from the product $[\ (g^3 + p^3 + r^3)(g^2 + p^2 + r^2)(g + p + r)\ ]$ where it is ensured that any assignment remains constant over a given team, as was required.

6 Polya's Fundamental theorem on Enumeration

Theorem 6.1 (Polya's fundamental enumeration theorem) The pattern inventory of a set of functions R^D acted upon by a permutation group G is obtained by replacing in the cycle index of the group, the variable x_i with the sum ${}^{\text{P}}{}_{r \in R}\, w(r)^i$. That is,

$$P.I. = P_G(\underset{r \in R}{\text{X}} w(r), \underset{r \in R}{\text{X}} w(r)^2, \dots, \underset{r \in R}{\text{X}} w(r)^n).$$

Proof: Some patterns have equal weight W_i. Let there be m_i such patterns. The set of functions of weight W_i is a union of disjoint subsets of $R^D, T = \bigcup_{i=1}^{m_i} F_i$.

Now G acts on T producing m_i orbits, so Burnside's Lemma gives

$$m_i = \frac{1}{|G|}\sum_{\pi\in G}|S^\pi|$$

where S^π is the subset of T fixed by π.

The pattern inventory is therefore given by

$$P.I. = \sum_i m_i W_i$$

$$= \frac{1}{|G|}\sum_{\pi\in G}\sum_i |S^\pi W_i|$$

$$= \frac{1}{|G|}\sum_{\pi\in G} |(W_f ix(\pi)|$$

where $W_f ix(\pi)$ is the total weight of all functions fixed by π.

Choose a $\pi \in G$ with cycle type $(b_1, b_2, \ldots, b_n)$. Any function in S^π, we have seen, is constant over the cycles of π, which together partition D. Thus, Lemma 5.5 gives

$$P.I. = \frac{1}{|G|}\sum_{\pi\in G}\prod_j \Big(\sum_{r\in R} w(r)^j\Big)^{b_j}$$

This is because we collect the similar expressions for the b_j cycles of equal length j. Comparing this with the definition

$$P_G(x_1, x_2, .., x_n) = \frac{1}{|G|}\sum_{\pi\in G}\prod_j x_j^{b_j}$$

the theorem is proved.

Corollary - The number of patterns is given by $P_G(|R|, |R|, \ldots, |R|)$.

7 Applications

Example 7.1 We return to the problem of vertex colouring of a cube in two colours, red and blue. The pattern inventory, by Polya's theorem, is

$$\frac{(r+b)^8 + 8(r+b)^2(r^3+b^3)^2 + 9(r^2+b^2)^4 + 6(r^4+b^4)^2}{24}$$

a) The number of patterns

$$= \frac{2^8 + 8.2^2.2^2 + 9.2^4 + 6.2^2}{24}$$

$$= 23.$$

b) The number of patterns of 4 vertices each coloured red and blue = coeffficientof r^4b^4 in the P.I.

$$= \frac{\binom{8}{4} + 8\cdot2\cdot2 + 9\cdot6 + 6\cdot2}{24}$$

$$= 7.$$

Example 7.2 How many necklaces of n beads can be made in m colours, if the group acting on them is the cyclic group C_n? We saw that the cycle index was given by

PG(*x*1,*x*2,..,*xn*) = X*φ*(*k*)*xn*/*kk*

k|*n*

and thus the number of necklaces, by Polya's theorem, is

N = X*φ*(*k*)*mn*/*k*.

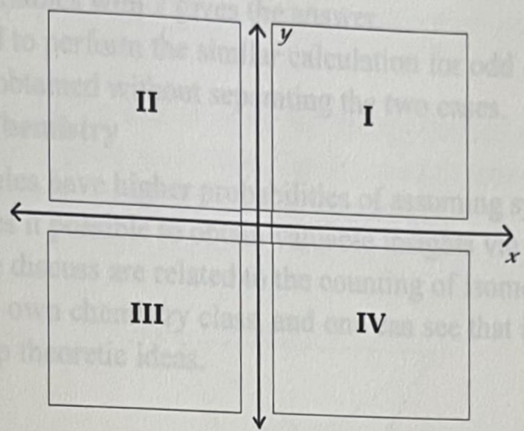
k|*n*

Remark: There is an elegant solution to the above problem by means of the Mobius Inversion Formula. The necklace must, as we showed, have cycle length *k*|*n*, and if the d repetitions due to permutations within each cycle are considered we have m^n = X*kM*(*k*)

k|*n*

where *M*(*k*) is the number of patterns with cycle length *k*. It can be proved that *M*(*k*) is multiplicative and thus the Inversion Formula can be applied, giving us the same answer after a lengthy computation which we will avoid here.

Example 7.3 We consider the action of the dihedral group D_4 on an $n \times n$ square chessboard where n is even. For the purpose of calculation we introduce a 'dummy' square at the origin and allow each unit square to have side 1 so that a lattice point is at their centre. The sides of the square have length 1. We first obtain the cycle index of the board.



a) Rotations by 90° or 270°: Each 1×1 square belongs to a subgroup of four squares under these motions and so for every four squares a 4-cycle is generated thus giving a monomial of $2\times x_4^{n^2/4}$. The 180° rotation gives $x_2^{n^2/2}$.

b) Here we assume that the motion of any unit square corresponds to the motionof the lattice point within the square. Consider the reflection about the xaxis followed by rotation through 90° or 270°. To analyse this we use the representation of points on R^2 as a 2-D coloumn vector and recall that the motions of D_n can be represented using orthogonal matrices A with $A^2 = I$. Thus any square will be part of a 2- cycle or less. It is now sufficient to find the number of fixed points.

Rotation by 90° and reflection about the x- axis have the associated matrices $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; and their product is $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ which operates on the vector $\begin{pmatrix}x\\ y\end{pmatrix}$ to give $\begin{pmatrix}y\\ x\end{pmatrix}$. Thus, under this operation, only the n squares along one diagonal are stabilized and the rest belong to 2- cycles. The monomial is the same for the cas of reflection and then 270° rotation, so we finally get the term

$2\cdot x_1^n x_2^{(n^2-n)/2}$. The remaining reflection is just reflection about the y- axis with the term $x_2^{n^2/2}$.

Our cycle index, then, is

$$\frac{x_1^{n^2} + 2x_4^{n^2/4} + 3x_2^{n^2/2} + 2x_1^n x_2^{(n^2-n)/2}}{8}$$

and replacing the variables with 2 gives the answer.

The reader is invited to perform the similar calculation for odd *n* and see if a closed form can be obtained without separating the two cases.

8 Applications to Chemistry

The fact that molecules have higher probabilities of assuming symmetrical configurations makes it possible to obtain valuable insights via group theory. All the problems we discuss are related to the counting of isomers. The first example is from my own chemistry class, and one can see that it can turn messy without group theoretic ideas.

Example 8.1 A *coordination complex* is a compound formed when a positively charged transition metal atom attracts negatively charged *ligands* in a solution. The number of ligands that coordinate with it determine the geometry of the complex.

Thus, six ligands form an octahedral complex written as *M(abcdef)* where M is the metal and the rest are ligands, some of which may be equal to each other. We are to find the number of isomers where *a,b,c,d,e,f* are all distinct. To solve this, we note that the octahedron is a dual of the cube in the sense that the vertices of the octahedron correspond to the faces of the cube, and vice versa. Therefore, the cycle index of the group of rotations actng on the vertices is exactly the same as that of the same group acting on the faces of the cube. and this was found in Example 5.4 to be

$$\frac{x_1^6 + 8x_3^2 + 6x_2^3 + 6x_1^2 x_4 + 3x_1^2 x_2^2}{24}.$$

By Polya's theorem, the number of diferent cubes obtained is the coefficient of *abcdef* in

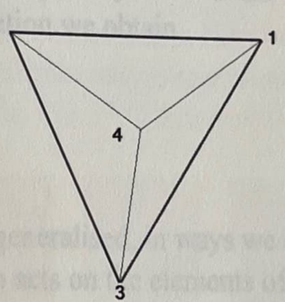
$$\frac{(a+b+c+d+e+f)^6}{24}$$

$$= \binom{6}{1,1,1,1,1,1}$$

$$= 30.$$

Note that this procedure counts both geometrical and optical isomers.

Example 8.2 There is a great deal of scope for the application of Polya Theory in the study of organic molecules. Carbon in its unsaturated form, is the centre of a tetrahedral molecule idealised as shown.



There are two rotations that fix the substituent 4, through 120˚ and 240˚, permuting the other 3 substituents in a 3- cycle. Thus, these rotations contribute a term $4 \cdot 2 \cdot x_1x_3$ to the cycle index. Next, we fix the axis on the line joining the midpoints of opposite sides, say 14 and 23. 180˚ rotations about this axis give us the cycles (14),(23). There are 3 such axes, giving us the term $3 \cdot x_2^2$. Including the identity, we obtain the cycle index

$$\frac{x_1^4 + 8x_1x_3 + 3x_2^2}{12}.$$

a) Existence of the enantiomeric form: Suppose the substituents *p,q,r,s* are all different. We want to find the number of molecules of the type *Cpqrs*. This

is the coefficient og *pqrs* in the pattern inventory, which is $\frac{\binom{4}{1,1,1,1}}{12} = 2.$

The occurrence of two patterns (non- superimposable molecules) when the substituents are different is called *chirality* and the molecules form anenamtiomeric pair.
These molecules form a mirror- image. Indeed, by fixing two substituents, we can interchange the other two, i.e. reflect them about the mirror formed by the plane of the fixed substituents. This also tells us why the substituents need to be different: otherwise, we can choose the mirror so that one molecule is reflected into itself.

b) A chemist studying the hydrogen content of alkyl halides with one carbonmust know how many compounds are possible for a given number of hydrogen substituents. This asks for the pattern inventory with H weighted with h and the halogens Cl, Br, I weighted with 1 each. The pattern inventory is obtained by substituting for x_i the quantity $h^i + 3$ in the cycle index, and carrying out the required computation we obtain

$$P.I. = h^4 + 3h^3 + 6h^2 + 11h + 5$$

which solves the problem.

This theory can be extended and generalised, in ways we shall not look into here; for example, what if a group acts on the elements of the range as well? A generalisation to Polya's theorem in this regard was provided by Nikolaas de Bruijn, and there are various extensions that I hope to be able to acquaint myself with in future.

I am grateful to Professor B. Sury of the Indian Statistical Institute, Bangalore, for guiding me and suggesting that I read some articles, including his own, related to Burnside's Lemma and related aspects of group theory, which became the motivation for this study.

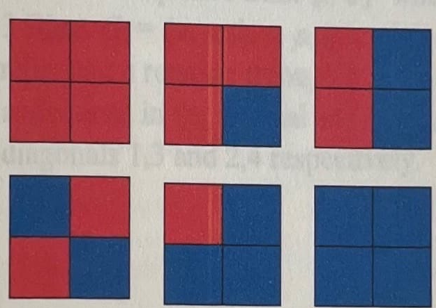
Polya's Theory of Counting

Example 1 A disc lies in a plane. Its centre is fixed but it is free to rotate. It has been divided into n sectors of angle $2\pi/n$. Each sector is to be colored Red or Blue. How many different colorings are there?

One could argue for 2^n.

On the other hand, what if we only distinguish colorings which cannot be obtained from one another by a rotation. For example if $n = 4$ and the sectors are numbered 0,1,2,3 in clockwise order around the disc, then there are only 6 ways of coloring the disc – 4R, 4B, 3R1B, 1R3B, RRBB and RBRB.

Now consider an $n \times n$ "chessboard" where $n \geq 2$. Here we color the squares Red and Blue and two colorings are different only if one cannot be obtained from another by a rotation or a reflection. For $n = 2$ there are 6 colorings.



The general scenario that we consider is as follows: We have a set X which will stand for the set of colorings when transformations are not allowed. (In example 1, $|X| = 2^n$ and in example 2, $|X| = 2^{n^2}$).

In addition there is a set G of permutations of X. This set will have a group structure:

Given two members $g_1, g_2 \in G$ we can define their composition $g_1 \circ g_2$ by $g_1 \circ g_2(x) = g_1(g_2(x))$ for $x \in X$. We require that G is *closed* under composiiton i.e. $g_1 \circ g_2 \in G$ if $g_1, g_2 \in G$.

We also have the following:

A1 The *identity* permutation $1_X \in G$.

A2 $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$ (Composition is associative).

A3 The inverse permutation $g^{-1} \in G$ for every $g \in G$.

(A set G with a binary relation $\circ$ which satisfies **A1,A2,A3** is called a **Group**).

In example 1 $D = \{0, 1, 2, \ldots, n-1\}$, $X = 2^D$ and the group is $G_1 = \{e_0, e_1, \ldots, e_{n-1}\}$ where $e_j * x = x + j \bmod n$ stands for rotation by $2j\pi/n$.

In example 2, $X = 2^{[n]2}$. We number the squares 1,2,3,4 in clockwise order starting at the upper left and represent X as a sequence from $\{r, b\}^4$ where for example rrbr means color 1,2,4 Red and 3 Blue. $G_2 = \{e, a, b, c, p, q, r, s\}$ is in a sense independent of n. e, a, b, c represent a rotation through 0, 90, 180, 270 degrees respectively. p, q represent reflections in the vertical and horizontal and r, s represent reflections in the diagonals 1,3 and 2,4 respectively.

	e	a	b	c	p	q	r	s
rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr	rrrr
brrr	brrr	rbrr	rrbr	rrrb	rbrr	rrrb	brrr	rrbr
rbrr	rbrr	rrbr	rrrb	brrr	brrr	rrbr	rrrb	rbrr
rrbr	rrbr	rrrb	brrr	rbrr	rrrb	rbrr	rrbr	brrr
rrrb	rrrb	brrr	rbrr	rrbr	rrbr	brrr	rbrr	rrrb
bbrr	bbrr	rbbr	rrbb	brrb	bbrr	rrbb	brrb	rbbr
rbbr	rbbr	rrbb	brrb	bbrr	brrb	rbbr	rrbb	bbrr
rrbb	rrbb	brrb	bbrr	rbbr	rrbb	bbrr	rbbr	brrb
brrb	brrb	bbrr	rbbr	rrbb	rbbr	brrb	bbrr	rrbb
rbrb	rbrb	brbr	rbrb	brbr	brbr	brbr	rbrb	rbrb
brbr	brbr	rbrb	brbr	rbrb	rbrb	rbrb	brbr	brbr
bbbr	bbbr	rbbb	brbb	bbrb	bbrb	rbbb	brbb	bbbr
bbrb	bbrb	bbbr	rbbb	brbb	bbbr	brbb	bbrb	rbbb
brbb	brbb	bbrb	bbbr	rbbb	brbb	bbrb	bbbr	brbb
rbbb	rbbb	brbb	bbrb	bbbr	brbb	bbbr	rbbb	bbrb
bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb	bbbb

From now on we will write $g * x$ in place of $g(x)$.
Orbits: If $x \in X$ then its orbit
$O_x = \{y \in X : \exists g \in G$ such that $g * x = y\}$.
Lemma 1 The orbits partition X.
Proof $x = 1_X * x$ and so $x \in O_x$ and so $X = \text{S}_{x \in X} O_x$.

Suppose now that $O_x \cap O_y 6= \emptyset$ i.e. $\exists g_1, g_2$ such that $g_1 * x = g_2 * y$. But then for any $g \in G$ we have

$$g * x = (g \circ (g_1^{-1} \circ g_2)) * y \in O_y$$

and so $O_x \subseteq O_y$. Similarly $O_y \subseteq O_x$. Thus $O_x = O_y$ whenever $O_x \cap O_y 6= \emptyset$. The two problems we started with are of the following form: Given a set X and a group of permutations *acting* on X, compute the number of orbits i.e. distinct colorings.

A subset H of G is called a *sub-group* of G if it satisfies *axioms* **A1,A2,A3** (with G replaced by H).

The *stabilizer* S_x of the element x is $\{g : g * x = x\}$. It is a sub-group of G.

A1: $1_X * x = x$.

A3: $g, h \in S_x$ implies $(g \circ h) * x = g * (h * x) = g * x = x$.

A2 holds for any subset.

Lemma 2
If $x \in X$ then $|O_x||S_x| = |G|$.

Proof Fix $x \in X$ and define an equivalence relation $\sim$ on G by

$$g_1 \sim g_2 \text{ if } g_1 * x = g_2 * x.$$

Let the equivalence classes be $A_1, A_2, \ldots, A_m$. We first argue that
$|A_i| = |S_x| \quad i = 1, 2, \ldots, m. \qquad (1)$

Fix i and $g \in A_i$. Then

$$h \in A_i \leftrightarrow g * x = h * x \leftrightarrow (g^{-1} \circ h) * x = x$$

$$\leftrightarrow (g^{-1} \circ h) \in S_x \leftrightarrow h \in g \circ S_x$$

where $g \circ S_x = \{g \circ \sigma : \sigma \in S_x\}$.
Thus $|A_i| = |g \circ S_x|$. But $|g \circ S_x| = |S_x|$ since if $\sigma_1, \sigma_2 \in S_x$ and $g \circ \sigma_1 = g \circ \sigma_2$ then $g^{-1} \circ$

$(g \circ \sigma_1) = (g^{-1} \circ g) \circ \sigma_1 = \sigma_1 = g^{-1} \circ (g \circ \sigma_2) = \sigma_2$.

This proves (1).

Finally, $m = |O_x|$ since there is a distinct equivalence class for each distinct $g * x$.

x	O_x	S_x	
rrrr	$\{rrrr\}$	G	
brrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	E
rbrr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	x
rrbr	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	a
rrrb	{brrr,rbrr,rrbr,rrrb}	$\{e_0\}$	m
bbrr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	p
rbbr	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	l
rrbb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	e
brrb	{bbrr,rbbr,rrbb,brrb}	$\{e_0\}$	
rbrb	{rbrb,brbr}	$\{e_0, e_2\}$	1
brbr	{rbrb,brbr}	$\{e_0, e_2\}$	
bbbr	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	$n = 4$
bbrb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
brbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
rbbb	{bbbr,rbbb,brbb,bbrb}	$\{e_0\}$	
bbbb	{bbbb}	G	

x	O_x	S_x	
rrrr	{e}	G	
brrr	{brrr,rbrr,rrbr,rrrb}	{e,r}	E
rbrr	{brrr,rbrr,rrbr,rrrb}	{e,s}	x
rrbr	{brrr,rbrr,rrbr,rrrb}	{e,r}	a
rrrb	{brrr,rbrr,rrbr,rrrb}	{e,s}	m
bbrr	{bbrr,rbbr,rrbb,brrb}	{e,p}	p
	{bbrr,rbbr,rrbb,brrb}	{e,q}	
	rbbr rrbb	{e,p}	
	{bbrr,rbbr,rrbb,brrb}		
brrb	{bbrr,rbbr,rrbb,brrb}	{e,q}	
rbrb	{rbrb,brbr}	{e,b,r,s}	2
brbr	{rbrb,brbr}	{e,b,r,s}	
bbbr	{bbbr,rbbb,brbb,bbrb}	{e,s}	
bbrb	{bbbr,rbbb,brbb,bbrb}	{e,r}	
brbb	{bbbr,rbbb,brbb,bbrb}	{e,s}	
rbbb	{bbbr,rbbb,brbb,bbrb}	{e,r}	
bbbb	{e}	G	

Let $\nu_{X,G}$ denote the number of orbits.

Theorem 1 $\nu_{X,G} = \frac{1}{|G|}\sum_{x\in X} |S_x|$.

Proof

$$\nu_{X,G} = \sum_{x\in X} \frac{1}{|O_x|} = \sum_{x\in X} \frac{|S_x|}{|G|},$$

from Lemma 1.

Thus in example 1 we have

$$\nu_{X,G} = \frac{1}{4}(4+1+1+1+1+1+1+1+1+1+2+2+1+1+1+1+4) = 6.$$

In example 2 we have

$$\nu_{X,G} = \frac{1}{8}(8+2+2+2+2+2+2+2+2+4+4+2+2+2+2+8) = 6.$$

Theorem 1 is hard to use if $|X|$ is large, even if $|G|$ is small.

For $g \in G$ let $Fix(g) = \{x \in X : g * x = x\}$.

Theorem 2

Frobenius, Burnside:

$$\nu X,^{G} = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

Proof Let $A(x, g) = 1_{g*x=x}$. Then

$$\nu X,^{G} = \frac{1}{|G|} \sum_{x \in X} |S_x|$$

1

$= \frac{}{}$ $A(x, g)$

$|G|$ X∈ X∈

x $X\, gG$

1

$= \frac{}{}$ $A(x, g)$

$|G|$ X∈ X∈

$gG\, xX$

1

$= \frac{}{}$ $|Fix(g)|$.

$|G|$ X∈

$g\, G$

Let us consider example 1 with $n = 6$. We compute

g	e_0	e_1	e_2	e_3	e_4	e_5
$\lvert Fix(g) \rvert$	64	2	4	8	4	2

Applying Theorem 2 we obtain

$$\nu_{X.G} = \frac{1}{6}(64 + 2 + 4 + 8 + 4 + 2) = 14$$

Cycles of a permutation:

Let $\pi : D \rightarrow D$ be a permutation of the finite set D. Consider the digraph $\Gamma_\pi =$ (D, A) where $A = \{(i, \pi(i)) : i \in D\}$. Γ_π is a collection of vertex disjoint cycles. Each $x \in D$ being on a unique cycle. Here a cycle can consist of a loop i.e. when $\pi(x) = x$.

Example: $D = [10]$.

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	2	7	10	3	8	9	1	5	4

The cycles are (1, 6, 8),(2),(3, 7, 9, 5),(4, 10).

In general consider the sequence $i, \pi(i), \pi^2(i), \ldots,$.

Since D is finite, there exists a first pair $k < \ell$ such that $\pi^k(i) = \pi^\ell(i)$. Now we must have $k = 0$, since otherwise putting $x = \pi^{k-1}(i)$ 6= $y = \pi^{\ell-1}(i)$ we see that $\pi(x) = \pi(y)$, contradicting the fact that π is a permutation.

So i lies on the cycle $C = (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i),\ i)$.

If j is not a vertex of C then $\pi(j)$ is not on C and so we can repeat the argument to show that the rest of D is partitioned into cycles.

Example 1

First consider $e_0, e_1, \ldots, e_{n-1}$ as permutations of D.

The cycles of e_0 are (1),(2),...,(n).

Now suppose that $0 < m < n$. Let $a_m = gcd(m, n)$ and $k_m = n/a_m$. The cycle C_i of e_m containing the element i is is $(i,\ i+m,\ i+2m,\ldots,\ i+(k_m-1)m)$ since n is a divisor $k_m m$ and not a divisor of $k'm$ for $k' < k_m$. In total, the cycles of e_m are C_0, $C_1,\ldots,\ C_{am}-1$.

This is because they are disjoint and together contain n elements. (If $i + rm = i' + r'm \bmod n$ then $(r-r')m + (i-i') = \ell n$. But $|i-i'| < a_m$ and so dividing by a_m we see that we must have $i = i'$.)

Next observe that if coloring x is fixed by e_m then elements on the same cycle C_i must be colored the same. Suppose for example that the color of $i + bm$ is different from the color of $i + (b + 1)m$, say Red versus Blue. Then in $e_m(x)$ the color of $i + (b+1)m$ will be Red and so $e_m(x)$ 6= x. Conversely, if elements on the same cycle of e_m have the same color then in $x \in Fix(e_m)$. This property is not peculiar to this example, as we will see.

Thus in this example we see that $|Fix(e_m)| = 2^{am}$ and then applying Theorem 2 we see that

$$\nu_{X,G} = \frac{1}{n}\sum_{m=0}^{n-1} 2^{gcd(m,n)}.$$

Example 2

It is straightforward to check that when n is even, we have

g	e	a	b	c	p	q	r	s
|Fix(g)|	2n2	2n2/4	2n2/2	2n2/4	2n2/2	2n2/2	2n(n+1)/2	2n(n+1)/2

For example, if we divide the chessboard into 4 $n/2 \times n/2$ sub-squares, numbered 1,2,3,4 then a coloring is in $Fix(a)$ iff each of these 4 sub-squares have colorings which are rotations of the coloring in square 1.

Polya's Theorem

We now extend the above analysis to answer questions like: How many *distinct* ways are there to color an 8×8 chessboard with 32 white squares and 32 black squares?

The scenario now consists of a set D (*Domain*, a set C (colors) and $X = \{x : D \rightarrow C\}$ is the set of colorings of D with the color set C. G is now a group of permutations of D.

We see first how to extend each permutation of D to a permutation of X. Suppose that $x \in X$ and $g \in G$ then we define $g * x$ by

$$g * x(d) = x(g^{-1}(d)) \quad \text{for all } d \in D.$$

Explanation: The color of d is the color of the element $g^{-1}(d)$ which is mapped to it by g.
Consider Example 1 with $n = 4$. Suppose that $g = e_1$ i.e. rotate clockwise by $\pi/2$ and $x(1) = b$, $x(2) = b$, $x(3) = r$, $x(4) = r$.
Then for example $g * x(1) = x(g^{-1}(1)) = x(4) = r$, as before.

Now associate a **weight** w_c with each $c \in C$.
If $x \in X$ then

$$W(x) = \text{Y}\in wx(d).$$

d D

Thus, if in Example 1 we let $w(r) = R$ and $w(b) = B$ and take $x(1) = b$, $x(2) = b$, $x(3) = r$, $x(4) = r$ then we will write $W(x) = B^2R^2$.

For $S \subseteq X$ we define the **inventory** of S to be $W(S) = \text{X}\in\ W(x)$.

$x\ S$

The problem we discuss now is to compute the **pattern inventory** $PI = W(S^*)$ where S^* contains one member of each

orbit of X under G.
For example, in the case of Example 2, with $n = 2$, we get $PI = R^4 + R^3B + 2R^2B^2$

$$+ RB^3 + B^4.$$

To see that the definition of PI makes sense we need to prove **Lemma 3** If x, y are in the same orbit of X then $W(x) = W(y)$.
Proof Suppose that $g * x = y$. Then

$$W(y) \ = \text{Y}\in wy(d)$$

$d\ D$

$= Y_{\in} wg{*}x(d)$

$d\ D$

$= Y\in\ wx(g^{-1}(d))$

D

d

$= Y\in wx(d))$ (3)

$d\ D$

$= W(x)$

Note, that we can go from (2) to (3) because as d runs over D, $g^{-1}(d)$ also runs over d. -Let $\Delta = |D|$. If $g \in G$ has k_i cycles of length i then we define

$$ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}.$$

The **Cycle Index Polynomial** of G, C_G is then defined to be

$$C_G(x_1, x_2, \ldots, x_\Delta) = \frac{1}{|G|} \sum_{g \in G} ct(g).$$

In Example 2 with $n = 2$ we have

g	e	a	b	c	p	q	r	s
ct(g)	$x_1 4$	x_4	$x_2 2$	x_4	$x_2 2$	$x_2 2$	$x12x2$	$x12x2$

and so

$$C_G(x_1, x_2, x_3\ x_4) = \frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4).$$

In Example 2 with $n = 3$ we have

g	e	a	b	c	p	q	r	s
ct(*g*)	$x_1$9	*x*1*x*42	*x*1*x*24	*x*1*x*42	$x_1$3$x_2$3	$x_1$3$x_2$3	$x_1$3$x_2$3	$x_1$3$x_2$3

and so

$$C_G(x_1\ x_2, x_3, x_4) = \frac{1}{8}(x_1^9 + x_1 x_2^4 + 4x_1^3 x_2^3 + 2x_1 x_4^2).$$

$PI = \sum m_i W_i$

$\sum_{i=1}^{m} W_i \left(\frac{1}{|G|} \sum_{g \in G} |Fix(g^{(i)})| \right)$ by Theorem 2

$= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} |Fix(g^{(i)})| W_i$

$$= \frac{1}{|G|} \sum_{g \in G} W(Fix(g)) \tag{4}$$

Note that (4) follows from $Fix(g) = \bigcup_{i=1}^{m} Fix(g^{(i)})$ since $x \in Fix(g^{(i)})$ iff $x \in X_i$ and $g * x = x$.

Suppose now that $ct(g) = x_1^{k_1} x_2^{k_2} \cdots x_\Delta^{k_\Delta}$ as above. Then we claim that

$$W(Fix(g)) = \left(\sum_{c \in C} w_c\right)^{k_1} \left(\sum_{c \in C} w_c^2\right)^{k_2} \cdots \left(\sum_{c \in C} w_c^\Delta\right)^{k_\Delta}. \tag{5}$$

Substituting (5) into (4) yields the theorem.

To verify (5) we use the fact that if $x \in Fix(g)$, then the elements of a cycle of g must be given the same color. A cycle of length i will then contribute a factor $\sum_{c \in C} w_c^i$ where the term w_c^i comes from the choice of color c for every element of the cycle.P

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1) Combinatorial Techniques, Sharad S. Sane (This was my main source for this topic)

2) Algebra, M. Artin (I learnt the basic group theory required from this book)

EULERIAN AND HAMILTONIAN GRAPH

A Project Submitted

To the

University of Mumbai

For the M.Sc. Degree

In Mathematics

Submitted by

Jaysawal Archita Shivkumar

Under the Guidance of

Prof. Dipali Mali (HoD)

Asst. Prof. Ishrat Gawandi

Department of Mathematics

Sonopant Dandekar Shikshan Mandali's

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Palghar – 401404

For The Year

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STATEMENT TO BE INCORPORATED BY THE CANDIDATE IN THE PROJECT AS REQUIRED UNDER REGULATION FOR THE M.A/M.SC. DEGREE

As required by University Regulation No. R 2190 we wish to state that the work embodied in the thesis titled "**Eulerian and Hamiltonian Graph**" forms our own. The Project of expository in nature and does not contain any new results, carried out underthe guidance of **Prof. Dipali Mali (HoD)** and **Asst. Prof. Ishrat Gawandi** at the Department of Mathematics, SDSM College. This work has not been submitted for any other degree of this or anyother Univer sity. Whenever reference have been made to previous works of others, it has been clearly indicated as such and included in the bibliography.

Place: SDSM College, Palghar,

Date:25 May 2023

Signature of Student

Jaysawal Archita Shivkumar

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We would like to express our special thanks of gratitude to "**Asst. Prof. Ishrat Gawandi**" for their able guidance and supportin completing our project.

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Date:25 May 2023

Place: SDSM college, palghar

Signature of Student

Jaysawal Archita Shivkumar

CERTIFICATE

This is to certify that **Jaysawal Archita Shivkumar** has worked under our supervision for Master of Science Project title **Eulerian and Hamiltonian Graph.** The Project is of expository in nature and does not contain any new results. Also the Project has not been submitted to this University for any other University for any other degree.

25/5/23

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DECLARATION

We declare that this written submission represent our Ideas in our own words and where other's Idea have been included, we have adequately cited and referenced the original sources. We also declare that we have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any of the ideas / data/ fact/ source in our submission. We understand that any violation of the above will cause disciplinary action by the Institute and can also evoke penl action fron the sources which thus not been properly cited or from whom proper permission has not been taken when needed.

Signature

Jaysawal Archita Shivkmar

Date: 25May 2023

Place: SDSM College

Contents

Abstract

The field of mathematics plays vital role in various fields. One of the important areas in mathematics is graph theory which is used in structural models. The Königsberg Bridge problem and Hamilton's game would give rise to two concepts in graph theory named after Euler and Hamilton. The study of Eulerian graphs was initiated in the 18th century, and that of Hamiltonian graphs in the 19th century. These graphs possess rich structure, and hence their study is a very fertile field of research for graph theory.

There are many games and puzzles which can be analysed by graph theoretic concepts.In fact, the two early discoveries which led to the existence of graphs arose from puzzles, namely, the Konigsberg Bridge Problem and Hamiltonian Game, and these puzzles also resulted in the special types of graphs, now called Eulerian graphs and Hamiltonian graphs. Due to the rich structure of these graphs, they find wide use both in research and application.

The existence of Euler and Hamiltonian graph make it easier to solve a real-life problem. During the time of pandemic "Covid-19", it is very essential for each one of us to be vaccinated. Vaccination is done in the hospitals by using Eulerian and Hamiltonian graphs not only to prevent people from infecting but also to increase the speed of vaccination.

Chapter 1

Introduction of Graphs

Graph is defined as an ordered pair of a non-set of vertices and a set of edges. G =(V(G), E(G)).

Here, V (G) is the set of vertices or nodes and E(G) is the set of edges connecting the vertices.

Vertex / Node

One of the points on which the graph is defined and which may be connected by graph edges.

From fig 1.1 the set of vertex $V = \{v_1, v_2, v_3, v_4, v_5\}$

Edge

An edge of a graph is one of the connection between the nodes or vertices of the graph. It can be connection from one vertex to next.

From fig 1.1 the set of edge $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

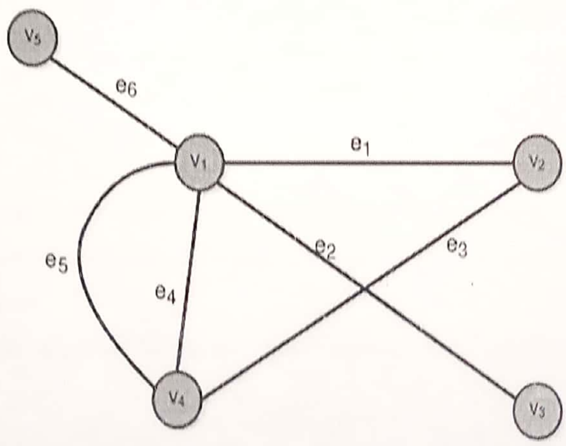

Fig 1.1 vertex and edge

Degree of a vertex

The degree (or valency) of a vertex of a graph is the number of edges that are incident to the vertex. It is denoted by $\delta(v)$ or deg(v) as shown in fig 1.2

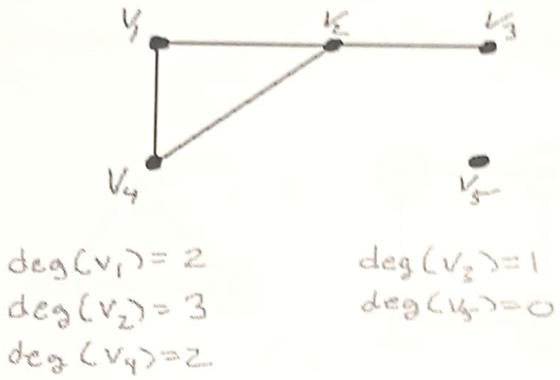

Fig 1.2 degree of vertex

Chapter 2

Types of Graph

2.1 Regular Graph

A graph in which degree of all the vertices is same is called as a regular graph.

If all the vertices in given graph are of degree 'k', then it is called as a "k-regular graph"

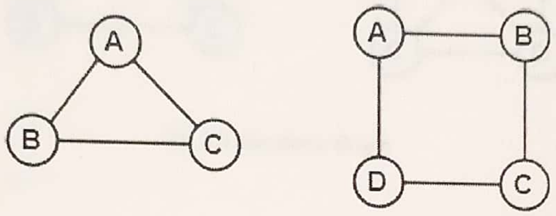

Fig 2.1 Regular Graph

In gigure 2.1, since each vertex in the graph is connected with all the remaining vertices through exactly one edge therefore, both graphs are complete graph.
In the above graph all the vertices have degree 2. Therefore it is 2- Regular graph.

2.2 Simple Graph

A simple graph is the undirected graph with no parallel edges and no loops.
In fig 2.2, First graph is not a simple graph because it has two edges between the vertices A and B and also has a loop.
Second graph is a simple graph because it does not contain any loop and parallel edges.

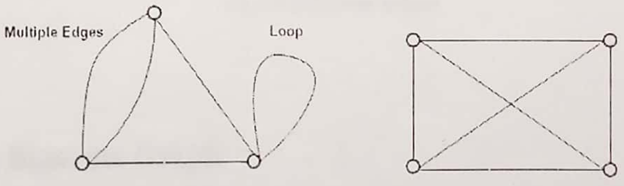

Fig 2.2 simple Graph

2.3 Complete Graph

A graph in which every pair of vertices is joined by exactly one edge is called complete graph. It contains all possible edges. It is denoted by K_n

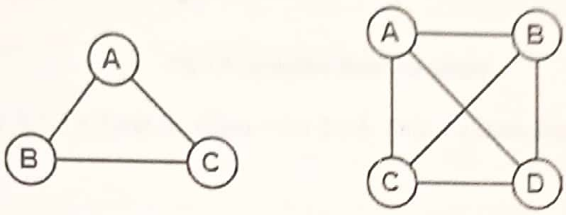

Fig 2.3 complete Graph

2.4 Bipartite Graph

A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V.

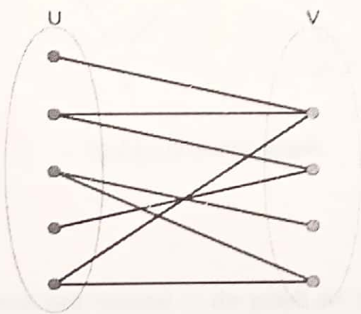

Fig 2.4 Bipartite Graph

2.5 Complete Bipartite Graph

A special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set.

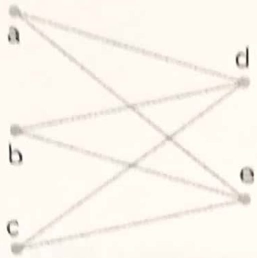

Fig 2.5 Complete Bipartite Graph

Here, total vertices are 3 + 2 = 5 and so, edges = 3 x 2 = 6. This is a characteristics of complete bipartite graph.

2.6 Connected Graph

A graph is said to be connected graph if there is a path between every pair of vertex. From every vertex to any other vertex there must be some path to traverse.

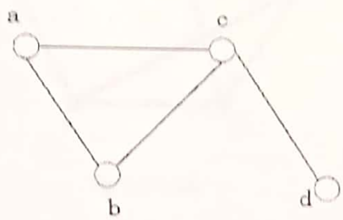

Fig 2.6 Connected Graph

2.7 Disconnected Graph

A graph is disconnected if at least two vertices of the graph are not connected by a path. If a graph G is disconnected, then every maximal connected subgraph of G is called a connected component of the graph G.

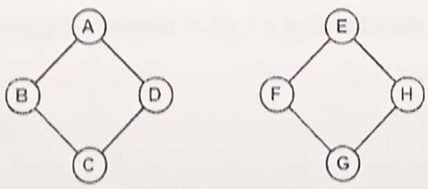

Fig 2.7 Disonnected Graph

Chapter 3

Basic Definitions

3.1 Walk

Definition: For a graph G=($V(G)$,$E(G)$), a Walk is defined as a sequence of alternating vertices and edges such as $v_0, e_1, v_1, e_2, \ldots .e_k, v_k$ where each edge e_i={v_{i-1}, v_i}. The Length of this walk is k.

For example, the graph fig 3.1 outlines a possibly walk (in blue). The walk is denoted as *abcdb*. Note that walks can have repeated edges. For example, if we had the walk *abcdcbce*, then that would be perfectly fine even though some edges are repeated.

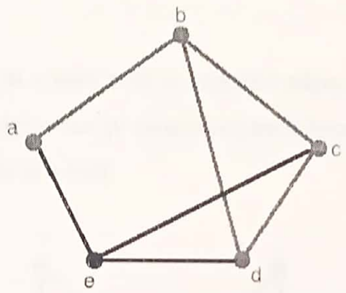

Fig 3.1 walk

Note that the length of a walk is simply the number of edges passed in that walk. In the graph above, the length of the walk is *abcdb* is 4 because it passes through 4 edges.

3.2 Open / Closed Walk

Definition: A walk is considered to be **Closed** if the starting vertex is the same as the ending vertex, that is $v_0 = v_k$. A walk is considered **Open** otherwise.

For example, the following graph shows in fig 3.2 a closed walk in green:

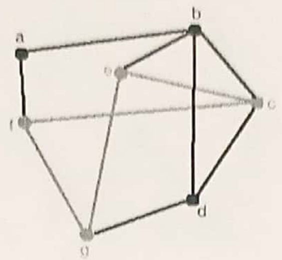

Fig 3.2 Open/closed walk

Notice that the walk can be defined by *cegfc*, and the start and end vertices of the walk is *c*. Hence this walk is closed.

3.3 Trail

Definition: A Trail is defined as a walk with no repeated edges.
So far, both of the earlier examples can be considered trails because there are no repeated edges. Here is another example of a trail:

Fig 3.3 Trail

Notice that the walk can be defined as *abc*. There are no repeated edges so this walk is also a trail.

3.4 Path

Definition: A **Path** is defined as an open trail with no repeated vertices.
Notice that all paths must therefore be open walks, as a path cannot both start and terminate at the same vertex. For example, in fig 3.4 orange coloured walk is a path.

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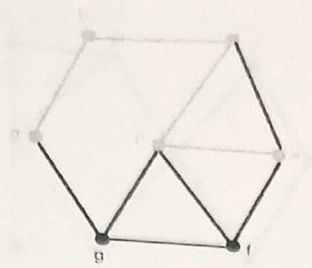

Fig 3.4 Path

Because the walk *abcde* does not repeat any edges.

3.5 Cycle

Definition: A Cycle is defined as a closed trail where no other vertices are repeated apart from the start/end vertex.

Fig 3.5 is an example of a circuit. Notice how no edges are repeated in the walk *bcgfb*, which makes it definitely a trail, and that the start and end vertex *b* is the same which makes it closed.

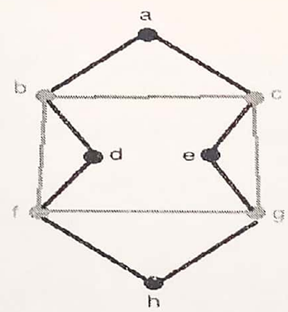

Fig 3.5 Cycle

3.6 Circuit

Definition: A Circuit is a closed trail. That is, a circuit has no repeated edges but may have repeated vertices.

An example of a circuit can be seen in fig 3.6. Notice how there are no edges repeated in the walk *hbcdefcgh*, hence the walk is certainly a trail. Additionally, the trail is closed, hence it is by definition a circuit.

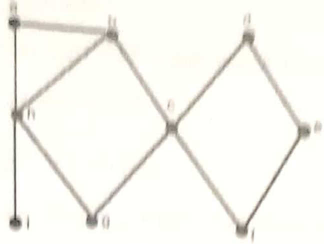

Fig 3.6 Circuit

Theorem 3.1 (Handshaking Lemma):

In any graph G $\sum_{v \in V} \deg(V) = 2|E|$ that is any graph G sum of degree of all the vertices is equal to two times of number of edges in it.

Proof: let $S = \{(v, e): v \in V(G), e \in E(G)\}$;

Count $|S|$ in two different ways. Fix $v \in V(G)$ then there are $\deg(V)$ possibilities for $e \in E$ such that $(v, e) \in S$

Hence, $|S| = \sum_{v \in V} \deg(V)$............(1)

On other hand if $e \in E(G)$ then there just two possibilities of $v \in V(G)$ such that $(v, e) \in S$

Hence, $|S| = 2|E|$................(2)

From (1) and (2),

$$\sum_{v \in V} \deg(V) = 2|E|$$

Hence, complete proof.

Theorem 3.2: In a finite singal graph number of vertices of odd degree is even.

Proof: Let G be a simple graph and Let A be set of even degree and Bvset of vertices of odd degree in G respectively.

Then $V(G) = A \cup B$ and $A \cap B = \emptyset$

Then by handshaking lemma we have,

$$2|E(G)| = \sum_{v \in V} \deg(V)$$

$$= \sum_{v \in A} \deg(V) + \sum_{v \in B} \deg(V)$$

Now, as $2|E(G)|$ and $\sum_{v \in A} \deg(V)$ both are even number.

Therefore, $\sum_{v \in B} \deg(V)$ is also an even number.

Chapter 4

Eulerian Graph

The story starts with the mayor of a Prussian city, who wrote to the famous mathematician **Leonhard Euler** with a question: how could one walk through Königsberg without crossing any of its bridges twice ? At first, Euler thought this question trivial, but the "Seven Bridges of Königsberg Problem" and its solution helped pave the way toward new mathematical branches of topology and graph theory.

4.1 Königsberg bridge problem / Seven bridge problem

The Konigsberg is the name of the German city, but this city is now in Russia.

In the below image, we can see the inner city of Konigsberg with the river Pregel.

There are a total of four land areas in which this river Pregel is divided, i.e., A, B, C and D.

There are total 7 bridges to travel from one part of the city to another part of the city.

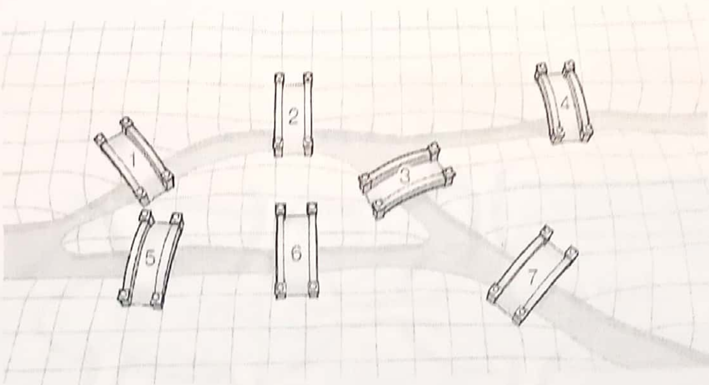

Fig 4.1 Konigsberg bridge

Solution of Konigsberg Bridge problem

In 1735, this problem was solved by Swiss mathematician Leon hard Euler.

According to the solution to this problem, these types of walks are not possible.

With the help of following graph, Euler shows the given solution.

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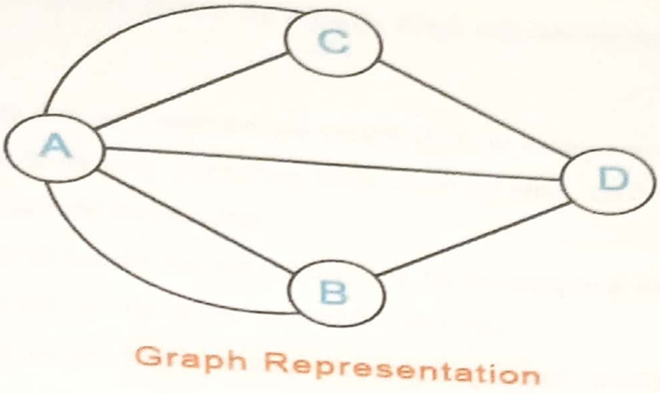


Fig 4.2 Konigsberg bridge simplified

In the above graph, the following things have:

The vertices of this graph are used to show the landmasses.

The edges are used to show the bridges.

Now we will consider an image and try to understand the concept of Konigsberg Bridge:

Fig 4.3

So, in the end, Euler proved that this type of walk is not possible. Euler proved it with the help of inventing a kind of diagram, which is known as the network.

In the above diagram, 3 bridges (arcs) were used to join riverbank A, and 3 arcs were used to join riverbank B.

Euler found that if a network contains the following things, only then that network will be traversable:

- The network does not contain any odd vertices. Here, the starting vertex and the end vertex must be the same. The starting vertex can be any vertex, but the ending point must be at the same starting vertex.
- Or the network contains two odd vertices. Here, the beginning point must be the one odd vertex, and the endpoint must be the other odd vertex.
- Since there are total 4 odd vertices in the Konigsberg network, therefore Euler concluded that the network is not traversable.
- Thus, Euler finally concluded that it is not possible to traverse the desired walking tour of Konigsberg.

Leonhard Euler

(1707-1783)

Leonhard Euler (15 April 1707 – 18 September 1783) was a Swiss mathematician, physicist, astronomer, logician and engineer who made important and influential discoveries in many branches of mathematics like infinitesimal calculus and graph theory while also making pioneering contributions to several branches such as topology and analytic number theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy, and music theory.

Euler was one of the most eminent mathematicians of the 18th century, and is held to be one of the greatest in history. He is also widely considered to be the most prolific mathematician of all time. His collected works fill 60 to 80 quarto volumes, more than anybody in the field. He spent most of his adult life in Saint Petersburg, Russia, and in Berlin, then the capital of Prussia.

A statement attributed to Pierre-Simon Laplace expresses Euler's influence on mathematics: "Read Euler, read Euler, he is the master of us all."

Fig 4.4 Leonhard Euler

4.2 Euler Path

A path in the connected graph that visits every edge of the graph exactly once with or without repeating the vertices, then such a path is called as an Euler path.

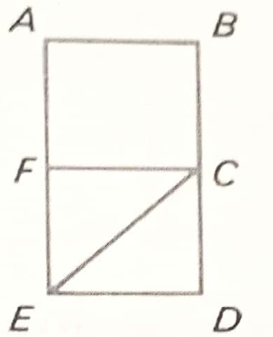


Fig 4.2 example of Euler Path

Euler path for fig 2.2 is F, A, B, C, F, E, C, D, E

This Euler path travels every edge once and only once and starts and ends at different vertices.

4.3 Euler Circuit

An Euler path that starts and ends at the same vertex is called as an Euler circuit.

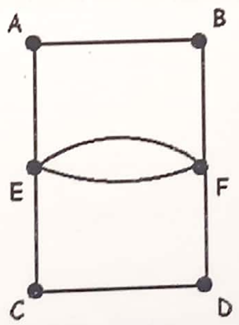


Fig 4.3 example of Euler Circuit

Euler circuit of above graph is E, A, B, F, E, F, D, C, E

This Euler path travels every edge once and only once and starts and ends at the same vertex.

Therefore, it is also an Euler circuit.

4.4 Eulerian Graph

A connected graph G is called an Euler graph, if there is a closed trail which includes every edge of the graph G. or An Euler Graph is a connected graph that contains an Euler Circuit.

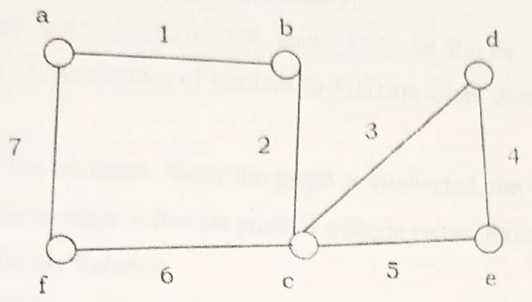


Fig 4.4 Example of Euler Graph

The above graph is an Euler graph as a 1 b2 c 3 d 4 e 5 c 6 f 7 g covers all the edges of the graph.

A closed walk in a graph G containing all the edges of G is called an *Euler line* in G. A graph containing an Euler line is called an *Euler graph*.

We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected except for any isolated vertices the graph may contain. As isolated vertices do not contribute anything to the understanding of an Euler graph, it is assumed now onwards that Euler graphs do not have any isolated vertices and are thus connected.

Theorem 4.1: (Euler's Theorem): A connected graph $G=(V(G),E(G))$ is Eulerian if and only if all vertices in $V(G)$ have an even degree.

Proof: We now have the necessities to prove Euler's theorem on Eulerian graphs.
We will prove this theorem using mathematical induction.
For a connected graph $G=(V(G),E(G))$, for each $e\geq 0$, let $S(e)$ be the statement that if G has e edges and all of the degrees of vertices in $V(G)$ are even, then the graph G is Eulerian.

Base Step: (e=0): $S(0)$ has no edges. Since the graph is connected, the only possibly way a connected graph can have no edges is that the graph is a single vertex, let's call it x_1. $\deg(x_1)=0$, which is even, and is trivially Eulerian.

Induction Step: $S(0)\wedge S(1)\wedge\ldots\wedge S(k-1)\Rightarrow S(k)$): Let $k\geq 1$ and assume that, $S(1),S(2),\ldots,S(k+1)$ is true. We want to prove $S(k)$ is value. Let G be a graph with k-edges, is connected, and all vertices of G have even degrees.

Since G is a connected graph, there are no isolated vertices, so it follows that the smallest degree $\delta(G)\geq 1$.
But all degrees are even, so $\delta(G)\geq 2$. From above, this graph G must contain a cycle, let's call it C.
Now let's create a new graph H by removing all of the edges that are in graph C from graph G. Note that the graph H may be disconnected. We can say the graph H is the union of the connected components $H_1, H_2 \ldots\ldots\ldots.. H_t$ The degree is each H_i must be even since the degrees drop only by 0 or 2.
Applying the induction hypothesis to each H_i that is $S(|E(H_1)|),\ldots,S(|E(H_t)|)$, each H_i will have an Eulerian circuit, let's say C_i.

We can now create a Eulerian circuit for G by splicing together the graph C with the Ci's. First start on any vertex of C_i and traverse until you hit some H_i. Then traverse C_i and continue back on C until you hit the next H_i.

Conclusion: Thus it follows that G must be Eulerian. This completes the inductive step as $S(0)\wedge S(1)\wedge\ldots\wedge S(k-1)\Rightarrow S(k)$). By the principle of strong mathematical induction, for $e\geq 0$, $S(e)$ is true.

Theorem 4.2: A connected graph G is Eulerian if and only if its edge set can be decomposed into cycles.

Proof : Let G(V, E) be a connected graph and let G be decomposed into cycles.

If k of these cycles are incident at a particular vertex v, then d(v) = 2k.

Therefore the degree of every vertex of G is even and hence G is Eulerian.

Conversely, let G be Eulerian.

We show G can be decomposed into cycles.

To prove this, we use induction on the number of edges.

Since d(v) ≥ 2 for each v ∈ V, G has a cycle C. Then G−E(C) is possibly a disconnected graph, each of whose components C1, C2, ..., Ck is an even degree graph and hence Eulerian.

By the induction hypothesis, each Ci is a disjoint union of cycles. These together with C provide a partition of E(G) into cycles.

Theorem 4.3: A connected graph is Eulerian if and only if each of its edges lies on an odd number of cycles.

Proof: *Necessity* Let G be a connected Eulerian graph and let e = uv be any edge of G.

Then G−e is a u−v walk W, and so G−e = W contains an odd number of u−v paths.

Thus each of the odd number of u−v paths in W together with e gives a cycle in G containing e and these are the only such cycles.

Therefore there are an odd number of cycles in G containing e.

Sufficiency Let G be a connected graph so that each of its edges lies on an odd number of cycles.

Let v be any vertex of G and $E_v = \{e_1, ..., e_d\}$ be the set of edges of G incident on v,

then $|E_v|$ = d(v) = d. For each i, 1 ≤ i ≤ d, let k_i be the number of cycles of G containing e_i.

By hypothesis, each k_i is odd.

Let c(v) be the number of cycles of G containing v.

Then clearly $c(v) = \frac{1}{2}\sum_{i=1}^{d} k_i$ implying that $2c(v) = \frac{1}{2}\sum_{i=1}^{d} k_i$.

Since 2c(v) is even and each k_i is odd, d is even.

Hence G is Eulerian.

Theorem 4.4: A connected graph G has an Eulerian trail if and only if it has at most two odd vertices.

i.e., it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof: Suppose G has an Eulerian trail which is not closed.

Since each vertex in the middle of the trail is associated with two edges

since there is only one edge associated with each end vertex of the trail, these end vertices must be odd and the other vertices must be even.

Conversely, suppose that G is connected with atmost two odd vertices.

If G has no odd vertices then G is Euler and so has Eulerian trail.

The leaves us to treat the case when G has two odd vertices (G cannot have just one odd vertex since in any graph there is an even number of vertices with odd degree).

Theorem 4.5: A connected graph G has an Euler walk if and only if exactly two vertices have odd degree.

Since a path may start and end at different vertices, the vertices where the path starts and ends are allowed to have odd degrees.

Proof. Suppos that G has an Euler walk starting at vertex v and ending at vertex w.

Add a new edge to the graph with endpoints v and w, forming G′.

G′ has an Euler circuit, and so by the previous theorem every vertex has even degree. The degrees of v and w in G are therefore odd, while all others are even.

Now suppose that the degrees of v and w in G are odd, while all other vertices have even degree.

Add a new edge e to the graph with endpoints v and w, forming G′.

Every vertex in G′ has even degree, so there is an Euler circuit which we can write as $v, e_1, v_2, e_2 \ldots, w, e, v$

so that $v, e_1, v_2, e_2 \ldots, w$ is an Euler walk.

Example

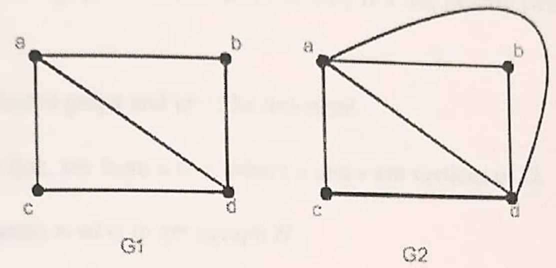


Fig 4.5 example of theorem 4.5

G_1 has two vertices of odd degree and and the rest of them have even degree. So this graph has an Euler path but not an Euler circuit. The path starts and ends at the vertices of odd degree. The path is a,c,d,a,b,d.

G_2 has four vertices all of even degree, so it has a Euler circuit. The circuit is a,d,b,a,c,d,a

Unicursal Graphs

An open walk that includes (or traces) all edges of a graph without retracing any edge is called a unicursal line or open Euler line. A connected graph that has a unicursal line is called a unicursal graph. Figure below shows a unicursal graph.

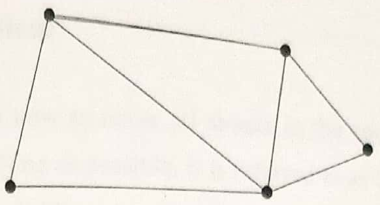


Fig 4.6 a unicursal graph.

Clearly by adding an edge between the initial and final vertices of a unicursal line, we get an Euler line.

Theorem 4.6: A connected graph is unicursal if and only if it has exactly two vertices of odd degree.

Proof: Let G be a connected graph and let G be unicursal.

Then G has a unicursal line, say from u to v, where u and v are vertices of G.

Join u and v to a new vertex w of G to get agraph H.

Then H has an Euler line and therefore each vertex of H is of even degree.

Now, by deleting the vertex w, the degree of vertices u and v each get reduced by one, so that u and v are of odd degree.

Conversely, let u and v be the only vertices of G with odd degree.

Join u and v to a new vertex w to get the graph H.

So every vertex of H is of even degree.

Thus H is Eulerian.Therefore, $G = H - w$ has a $u - v$ unicursal line so that G is unicursal

Sub Eulerian Graph: A graph G is said to be sub-Eulerian if it is a spanning subgraph of some Eulerian graph.

Supper Eulerian Graph: A non-Eulerian graph G is said to be *super-Eulerian* if it has a spanning Eulerian subgraph.

4.5 Chinese Postman Problem

The postman problem is how to cover all streets in the route and return back to the starting point with as short travelling as possible. It is referred to as the Chinese postman problem (CPP)because the first work publisged on problem was by Chinese mathematician Meigu Guan, in 962.

It says that a postman picks up mails at the post office, delivers it along a set of streets, and returns to the post office. In practice, apart from the requirement of traveling all streets, we consider the street direction, number of postmen. But in the present environment of consideration of a time window constraint is necessary. Chinese postman problem with time window constraint such that this problem can simulate the real situations.

Chinese Postman problem is a variation of Eulerian circuit problem for undirected graphs. An Euler Circuit is a closed walk that covers every edge once starting and ending position is same. Defined for connected and undirected graph. The problem is to find shortest path that visits every edge of the graph at least once. In order to do so, he (or she) must pass each street once and then return to the origin.

If input graph contains Euler Circuit, then a solution of the problem is Euler Circuit
An undirected and connected graph has Eulerian cycle if "all vertices have even degree".
If input graph contains Euler Circuit, then a solution of the problem is Euler Circuit.

Steps involve in solving Chinese postman problem:

1. Total weight = sum of weight of all edges

2. Identify all the odd vertices in the graph.

3. Determine all possible pairings of odd vertices.

4. Choose the pair with minimum weight.

5. add the combination of pairings found previously as edges to the original graph.

6. Identify new path.

We will apply the above steps in the following example to find a minimum Chinese post-man route.

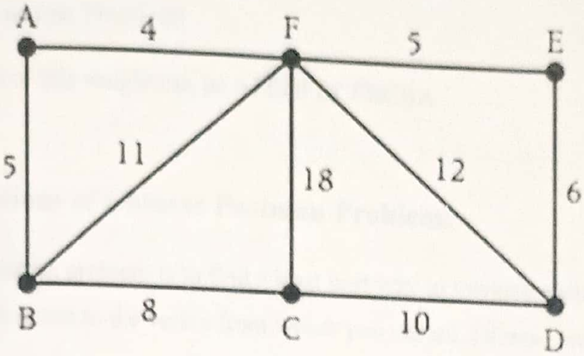


Fig 4.7 example of Chinese postman problem

B, C, F and D are odd vertices.

Possible pairings of odd vertices are BC-FD, BF-CD and BD-CF.

For each pairing find the edges that connect the vertices with the minimum weight:

BC-FD: 8+12=20.

BF-CD: 11+10=21.

BD-CF: 18+18=36

Find the pairings such that the sum of the weights is minimized: BC-FD

On the original graph add the edges that have been found in Step 4.

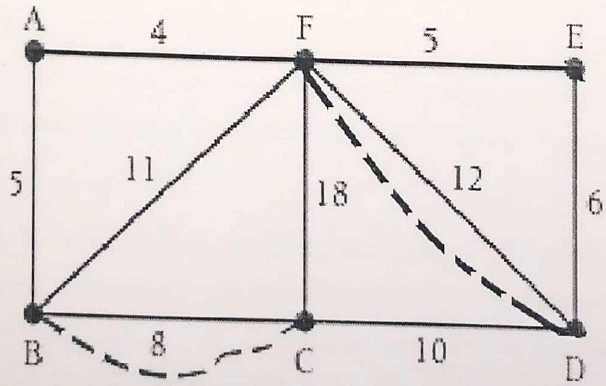


Fig 4.8 optimal graph of Chinese postman problem

Conclusion:

Total weight should be 79+20=99

A possible route of this weight can be AFEDFDCFBCBA

4.5.1 Applications of Chinese Postman Problem:

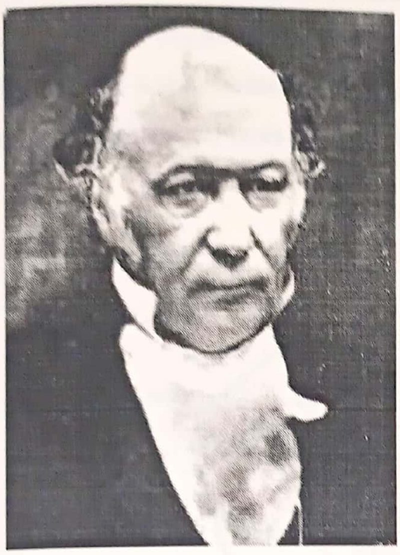
The Chinese postman problem is to find a least cost way to traverse each arc of a network at least once and to return to the vertex from which you started. Diverse problems such as

- The routing of road crews
- Police patrol scheduling
- Garbage collection
- Road sweeping
- School bus routing
- Transmission line inspections etc. can be done by Chinese postman problems.
- The practical example is planning of bus routing. In order to save the cost on the fuel, the bus company have modelled the bus stop as the vertix and the road as the edge in the bus route, then using the graph theory to obtain the optimal route that can meet the target of using the minimal fuel but across every road, at least once.

Chapter 5

Hamiltonian Graph

Sir William Rowan Hamilton

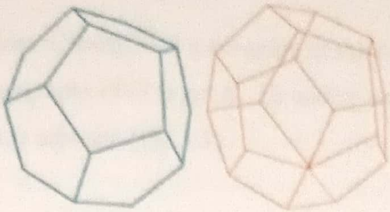


- **Sir William Rowan Hamilton** (August 4, 1805 – September 2, 1865) was an Irish mathematicianSir, physicist, and astronomer who made important contributions to the development of optics, dynamics, and algebra. His discovery of quaternions is perhaps his best known investigation. Hamilton's work was also significant in the later development of quantum mechanics.

Fig 5.1 William Rowan Hamilton

The cycle was named after Sir William Rowan Hamilton an Irish Mathematician (1805–1865) in 1857, invented a puzzle-game now also known as Hamilton's puzzle, which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron. Which involved hunting for a Hamiltonian cycle. The game, called the Icosian game, was distributed as a do-decahedron graph with a hole at each vertex. To solve the puzzle or win the game one had to use pegs and string to find the Hamiltonian cycle a closed loop that visited every hole exactly once.

The aim of the game was to construct, using the edges of the dodecahedron a close walk of all the cities which traversed each city exactly once, beginning and ending at the same city. In other word, one had essentially to form a Hamiltonian cycle in the graph corresponding to the dodecahedron fig. below shows such a cycle.



He labelled each of the vertices with the name of an important city. The challenge was to find a route along the edges of the dodecahedron which visited every city exactly once and returned to the start. Here is a graph which represents the dodecahedron. Here each of the 20 vertices, 30 edges and 12 pentagonal faces is represented in the graph.

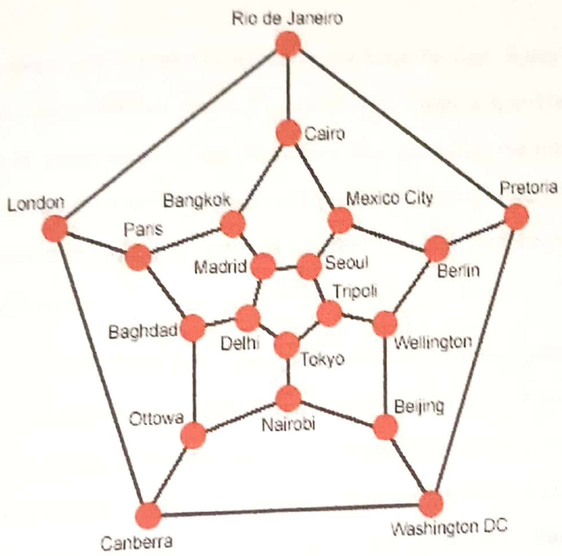


Fig 5.2 Dodecahedron

Hamilton solved this problem using the icosian calculus, an algebraic structure based on roots of unity with many similarities to the quaternions (also invented by Hamilton). This solution does not generalize to arbitrary graphs.

Clearly, the n-cycle C_n with n distinct vertices (and n edges) is Hamiltonian. Now, givenany Hamiltonian graph G, the supergraph G' (obtained by adding in new edges between non adja-cent vertices of G) is also Hamiltonian. This is because any Hamiltonian cycle in G is also a Hamiltonian cycle of G'. For instance, K_n is a supergraph of an n-cycle and so K_n is Hamilto-nian.

Let G be a graph with n vertices. Clearly, G is a subgraph of the complete graph K_n. From G, we construct step by step supergraphs of G to get K_n, by adding an edge at each step between two vertices that are not already adjacent (fig. 5.3).

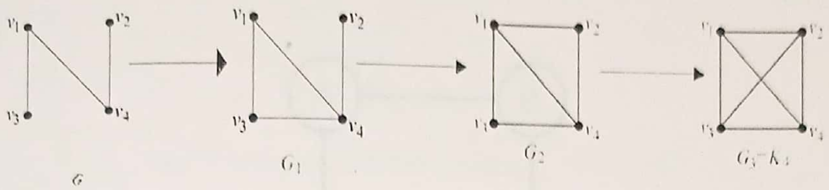


Fig 5.3 supergraphs of G to get K_n

Now, let us start with a graph G which is not Hamiltonian. Since the final outcome of the procedure is the Hamiltonian graph K_n, we change from a non-Hamiltonian graph toa Hamiltonian graph at some stage of the procedure. For example, the non-Hamiltonian graph G_1 above is followed by the Hamiltonian graph G_2. Since supergraphs of Hamiltonian graphs are Hamiltonian, once a Hamiltonian graph is reached in the procedure, all the subsequent supergraphs are Hamiltonian.

The main thing we'll need to be able to do with Hamiltonian graphs is decide whether a given graph is Hamiltonian or not. Although the definition of Hamiltonian graph is very similar to that of Eulerian graph, it turns out the two concepts behave very differently. While Euler's Theorem gave us a very easy criterion to check to see whether or not a graph Eulerian, there is no such criterion to see if a graph is Hamiltonian or not. It turns out that deciding whether or not a graph is Hamiltonian is NP-complete, meaning that if we could solve that problem efficiently, then you could solve a host of other difficult problems efficiently as well.

It may seem unfair, then, to ask whether a graph is Hamiltonian or not. But it's only in a very theoretical way that the problem is extremely difficult -- as the number of vertices get very large, the problem gets harder and harderquickly. For any given graph with a low number of vertices, there aren't that many possibilitie.

5.1 Hamiltonian Path

In a connected graph, if there is a walk that passes each and every vertex of a graph only once, this walk will be known as the Hamiltonian path. In this walk, the edges should not be repeated.

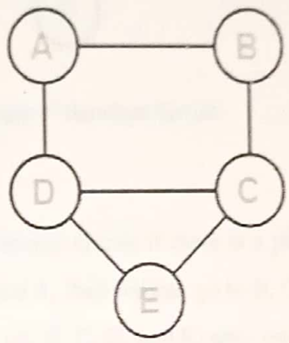


Fig 5.4 Example of Hamilton Path

In the above graph, we can see that when we start from A, then we can go to B, C, D, and then E. So this is the path that contains all the vertices (A, B, C, D, and E) only once, and there is no repeating edge. That's why we can say that this graph has a Hamiltonian path, which is described as follows:

Hamiltonian path = ABCDE

5.2 Hamiltonian Circuit

In a connected graph, if there is a walk that passes each and every vertex of the graph only once and after completing the walk, return to the starting vertex, then this type of walk will be known as a Hamiltonian circuit. For the Hamiltonian circuit, there must be no repeated edges. We can also be called Hamiltonian circuit as the Hamiltonian cycle.

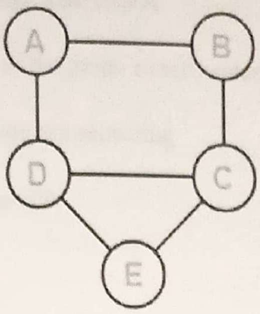


Fig 5.5 Example of Hamilton Circuit

The above graph contains the Hamiltonian circuit if there is a path that starts and ends at the same vertex. So when we start from the A, then we can go to B, C, E, D, and then A. So this is the path that contains all the vertices (A, B, C, D, and E) only once, except the starting vertex, and there is no repeating edge. That's why we can say that this graph has a Hamiltonian circuit.

Hamiltonian Circuit = ABCEDA

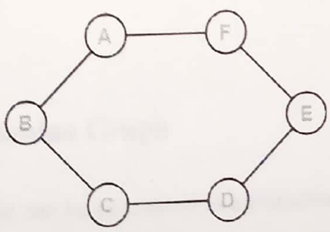
5.3 Hamiltonian Graph

The graph will be known as a Hamiltonian graph If there exists a closed walk in the connected graph that visits each and every vertex of the graph exactly once (except starting vertex) without repeating the edges,

OR

Any connected graph that contains a Hamiltonian circuit is called as a Hamiltonian Graph.

The following graph is an example of a Hamiltonian graph.



Example of Hamiltonian Graph

Fig 5.6

This graph contains a closed walk ABCDEFA.

It passed through every vertex of the graph exactly once except starting vertex.

At the time of walk, the edges are not repeating

Therefore, it is a Hamiltonian graph.

NOTE

If we remove one of the edges of Hamiltonian path, then it will be converted into any Hamiltonian circuit.

If any graph has a Hamiltonian circuit, then it will also have the Hamiltonian path. But the vice versa in this case will not be possible.

A graph can contain more than one Hamiltonian path and Hamiltonian circuit.

However, a non-Hamiltonian graph can have a Hamiltonian path, that is, Hamiltonian paths cannot always be used to form Hamiltonian cycles. For example, in Fig 5.7 G_1 has no Hamiltonian path, and so no Hamiltonian cycle; G_2 has the Hamiltonian path $v_1v_2v_3v_4$, but has no Hamiltonian cycle, while G_3 has the Hamiltonian cycle $v_1v_2v_3v_4v_1$.

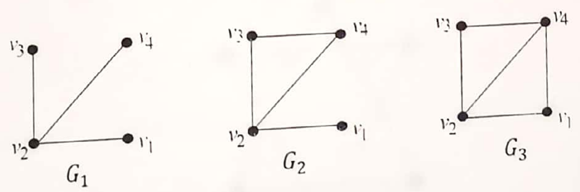


Fig 5.7

5.4 Properties of Hamiltonian Graph

- Any Hamiltonian cycle can be converted to a Hamiltonian path by removing one of its edges, but a Hamiltonian path can be extended to Hamiltonian cycle only if its end-points are adjacent.

- All Hamiltonian graphs are biconnected, but a biconnected graph need not be Hamiltonian.
- An Eulerian graph G (a connected graph in which every vertex has even degree) necessarily has an Euler tour, a closed walk passing through each edge of G exactly once. This tour corresponds to a Hamiltonian cycle in the line graph $L(G)$, so the line graph of every Eulerian graph is Hamiltonian. Line graphs may have other Hamiltonian cycles that do not correspond to Euler tours, and in particular the line graph $L(G)$ of every Hamiltonian graph G is itself Hamiltonian, regardless of whether the graph G is Eulerian.
- A tournament (with more than two vertices) is Hamiltonian if and only if it is strongly connected.
- The number of different Hamiltonian cycles in a complete undirected graph on n vertices is $\frac{(n-1)!}{2}$ and in a complete directed graph on n vertices is $(n-1)!$. These counts assume that cycles that are the same apart from their starting point are not counted separately.

Definition: A simple graph G is called *maximal non-Hamiltonian* if it is not Hamiltonian and the addition of an edge between any two non-adjacent vertices of it forms a Hamiltonian graph. For example Figure below shows a maximal non-Hamiltonian graph.

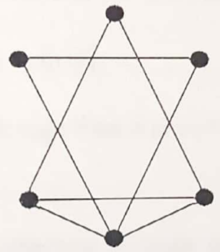


Fig 5.8 maximal non-Hamiltonian

It follows from the above procedure that any non-Hamiltonian graph with n-vertices is a subgraph of a maximal non-Hamiltonian graph with n vertices.

The above procedure is used to prove the following sufficient conditions due to Dirac.

Theorem 5.1 [Dirac (1952)] : If G is a simple graph having at least three vertices and $\deg(v) \geq m/2$, for every vertex v of G, then G is a Hamiltonian graph.

Proof: Assume that the result is not true. Then for some value $m \geq 3$, there is a non- Hamiltonian graph H in which $\deg(v) \geq m/2$, for every vertex of H.

In any spanning super graph K (i.e., with the same vertex set) of H, $\deg(v) \geq m/2$ for every vertex of K,

Since any proper supergraph of this form is obtained by adding more edges.

Thus there is a maximal non-Hamiltonian graph G with m vertices and $\deg(v) \geq m/2$ for every v in G. Using this G, we obtain a contradiction.

Clearly, G is not complete as K_n is Hamiltonian.

Therefore there are non-adjacent vertices u and v in G.

Let $G+uv$ be the supergraph of G by adding an edge between u and v.

Since G is maximal non-Hamiltonian, $G+uv$ is Hamiltonian.

Also, if C is a Hamiltonian cycle of $G+uv$, then C contains the edge uv, since otherwise C is a Hamiltonian cycle of G, which is not possible.

Let this Hamiltonian cycle C be $u = v_1, v_2, \ldots, v_n = v, u$.

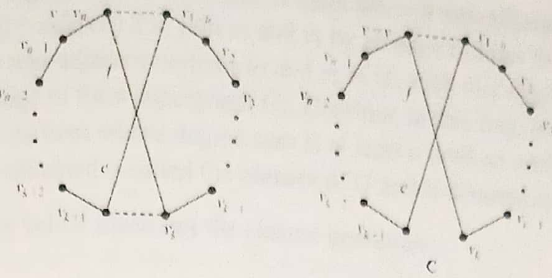
Now, let $S = \{v_i \in C$: there is an edge from u to v_{i+1} in $G\}$ and

$T = \{v_j \in C$: there is an edge from v to v_j in $G\}$.

Then $v_n \notin T$, since otherwise there is an edge from v to $v_n = v$, that is a loop, which is impossible.

Also $v_n \notin S$, (taking v_{n+1} as v_1), since otherwise we again get a loop from u to $v_1 = u$.

Therefore, $v_n \notin S \cup T$ (Fig. below).



Let $|S|$, $|T|$ and $|S \cup T|$ be the number of elements in S, T and $S \cup T$ respectively.

So $|S \cup T| < n$. Also, for every edge incident with u, there corresponds one vertex v_i in S.

Therefore, $|S| = deg(u)$. Similarly, $|T| = deg(v)$.

Now, if v_k is a vertex belonging to both S and T, there is an edge e joining u to v_{k+1} and an edge f joining v to v_k.

This implies that $C^{\prime} = v_1, v_{k+1}, v_{k+2}, \ldots, v_n, v_k, v_{k-1}, \ldots, v_2, v_1$ isa Hamiltonian cycle in G, which is a contradiction as G is non-Hamiltonian.

This shows that there is no vertex v_k in $S \cap T$, so that $S \cap T = \Phi$.

Thus $|S \cup T| = |S| + |T| - |S \cap T|$ gives $|S| + |T| = |S \cup T|$.

so that $deg(u) + deg(v) < n$.

This is a contradiction, because $deg(u) \geq m/2$ for all u in G, and so $deg(u) + deg(v) \geq m/2 + m/2$ giving $deg(u)+deg(v) \geq m$. Hence complete the proof.

Theorem 5.2 (Ore) Let G be a graph with n vertices and let u and v be non-adjacent vertices in G such that $deg(u)+deg(v) \geq n$. Let $G+uv$ denote the super graph of G obtained by joining u and v by an edge. Then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

Proof: Let G be a graph with n vertices and suppose u and v are non-adjacent vertices in G such that $deg(u) + deg(v) \geq n$.

Let $G + uv$ be the super graph of G obtained by adding the edge uv.

Let G be Hamiltonian. Then obviously $G + uv$ is Hamiltonian.

Conversely, let $G + uv$ be Hamiltonian.

We have to show that G is Hamiltonian.

Then, if G is not Hamitonian as in Theorem Dirac, we get inequality $deg(u) + deg(v) < n$, which contradicts the hypothesis that $deg(u) + deg(v) \geq n$. Hence G must be Hamiltonian.

Definition: Let G be a graph with n vertices. If there are two non-adjacent vertices u_1 and v_1 in G such that $deg(u_1) + deg(v_1) \geq n$, join u_1 and v_1 by an edge to form the super graph G_1. Now, if there are two non-adjacent vertices u_2 and v_2 in G_1 such that $deg(u_2) + deg(v_2) \geq n$, join u_2 and v_2 by an edge to form supergraph G_2. Continue in this way, recursively joining pairs of non-adjacent vertices whose degree sum is at least n until no such pair remains. The final supergraph thus obtained is called the ***closure*** of G and is denoted by $c(G)$.

The example in Figure below illustrates the closure operation

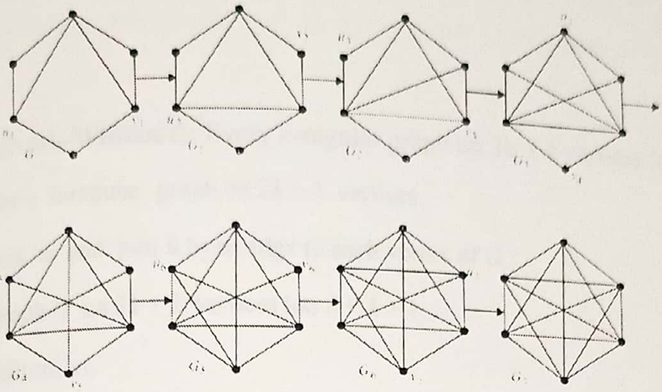
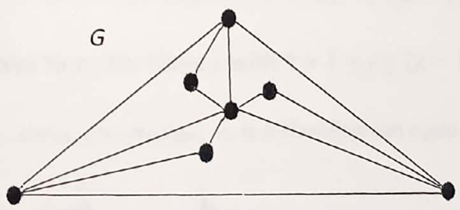


Fig 5.9 Closure

We observe in this example that there are different choices of pairs of non-adjacent vertices u and v with $deg(u)+deg(v) \geq n$. Therefore the closure procedure can be carried out inseveral different ways and each different way gives the same result.

In the graph shown in Figure 3.16, $n = 7$ and $deg(u)+deg(v) < 7$, for any pair u, v of adjacent vertices. Therefore, $c(G) = G$.



Theorem 5.3 (Bondy and Chvatal): A simple graph G is Hamiltonian if and only if its closure $c(G)$ is Hamiltonian

Proof: Let $c(G)$ be the closure of the graph G.

Since $c(G)$ is a supergraph of G.

Therefore, if G is Hamiltonian, then $c(G)$ is also Hamiltonian.

Conversely, let $c(G)$ be Hamiltonian.

Let $G, G_1, G_2, \ldots, G_{k-1}, G_k = c(G)$ be the sequenceof graphs obtained by performing the closure procedure on G.

Since $c(G) = G_k$ is obtained from G_{k-1} by setting $G_k = G_{k-1} + uv$, where u, v is a pair of non adjacent vertices in G_{k-1} with $d(u)+d(v) \geq n$,

therefore it follows that G_{k-1} is Hamiltonian.

Similarly G_{k-2}, so $G_{k-3}, \ldots, G_1$ and thus G is Hamiltonian.

Theorem 5.4 (Nash-Williams): Every k-regular graph on 2k + 1 vertices is Hamiltonian.

Proof: Let G be a k-regular graph on $2k+1$ vertices.

Add a new vertex w and join it byan edge to each vertex of G.

The resulting graph H on $2k+2$ vertices has $\delta = k+1$.

Thus H is Hamiltonian.

Removing w from H, we get a Hamiltonian path, say $v_0 v_1 \ldots v_{2k}$.

Assume that G is not Hamiltonian, so that

(a) if $v_0 v_i \in E$, then $v_{i-1} v_{2k} \notin E$

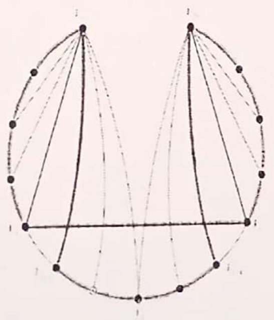
(b) if $v_0 v_i \notin E$, then $v_{i-1} v_{2k} \in E$, since $d(v_0) = d(v_{2k}) = k$.

The following cases arise.

Case (i) v_0 is adjacent to $v_1, v_2, \ldots, v_k$, and v_{2k} is adjacent to $v_k, v_{k+1}, \ldots, v_{2k-1}$. Then there is an i with $1 \leq i \leq k$ such that v_i is not adjacent to some v_j for $0 \leq j \leq k (j \neq i)$.

But $deg(v_i) = k$. So v_i is adjacent to v_j for some j with $k+1 \leq j \leq 2k-1$.

Then the cycle C given by $v_i v_{i-1} \ldots v_0 v_{i+1} \ldots v_{j-1} v_{2k} v_{2k+1}\ v_j$ is a Hamiltonian cycle of G (Fig below).



Case (ii) There is an i with $1 \le i \le 2k-1$ such that $v_{i+1}v_0 \in E$, but $v_iv_0 \notin E$.

Then by (b), $v_{i-1}v_{2k} \in E$. Thus G contains the $2k$-cycle $v_{i-1}v_{i-2} \ldots v_0v_{i+1}$.

Renaming the $2k$-cycle C as $u_1u_2 \ldots u_{2k}$ and let u_0 be the vertex of G not on C.

Then u_0 cannot be adjacent to two consecutive vertices on C and hence u_0 is adjacent to every second vertex on C, say $u_1, u_3, \ldots, u_{2k-1}$.

Replacing u_{2i} by u_0, we obtain another maximum cycle C' of G.

Hence u_{2i} must be adjacent to $u_1, u_3, \ldots, u_{2k-1}$. But then u_1 is adjacent to $u_0, u_2, \ldots, u_{2k}$, implying

$deg(u_1) \ge k+1$. This is a contradiction.

Hence G is Hamiltonian.

5.5 Travelling Salesman Problem

The origin of travelling salesman problem is not clear. A handbook for travelling salesman from 1832 mentions the problem and includes example tours through Germany and Switzerland, but contains no mathematical treatment. The problem was first formulated in 1930 and is one of the most intensively studied problems in optimization. The TSP was mathematically formulated in 19th century by the Irish mathematician William Rowan Hamilton and by British mathematician Thomas Kirkman. Hamilton's icosian game was a recreational puzzle based on finding a Hamiltonian cycle. The general form of the TSP appears to have been first studied by mathematicians during the 1930.

This problem is related to Hamiltonian cycle , Salesman is required to visit number of cities during the trip with covering least possible total distance. In which we use given cities as a vertex, different road between cities as a edges and distance between cities as a weight of that edge between vertices. For this equivalent concept in graph theory is require to find a Hamiltonian cycle of least possible total weight in a weighed graph.

A salesman travels from city to city with in his alloted territory in such a manner that he covers all the cities once and only once during his tour and come back to his base city. The travelling salesman problem can be solved as an assignment problem with following restrictions.

- The salesman starts his journey from his base city and come back to his base city after visiting all the cities once and only once during tour alloted to him.
- The objectives of salesman is to cover minimum distance.
- The salesman can not travel from a city to same city. Although the cost of travelling city to same city zero, but this assignment is prohibited.

Illustration

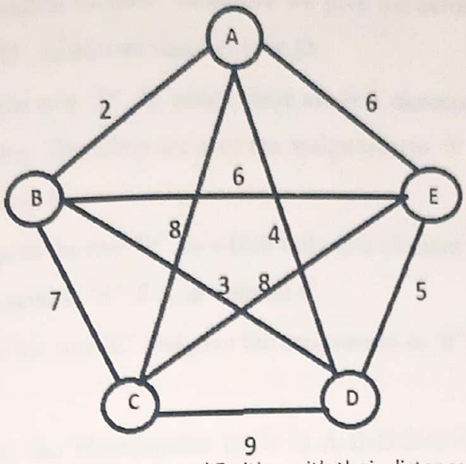


Fig. 5.10 Graph represented 5 cities with their distance

Let's suppose A,B,C,D & E are 5 cities and there are many routes from one city to other city. Suppose to find shortest route from city A to A crossing all remaining cities.

Given data in the graph represented as.

	A	B	C	D	E
A	0	2	8	4	6
B	2	0	7	3	6
C	8	7	0	9	8
D	4	3	9	0	5
E	6	6	8	5	9

Now let's give the assignment in above table such that each row or column has only one assignment.

- First we avoid column 'A' because in this column give the assignment in the last. Now go to the 1st row 'A' in which there are four elements 2,8,4 and 6 out of which 2 is a smallest number. Therefore we give the assignment to '2'. (Remember that assignment completed column is avoid for the next assignment) . Therefore we avoid column 'B'. minimum number is in B.

- Now next we go to the row 'B'. In which there are three elements 7,3 and 6 out of which 3 is a smallest number. Therefore we give the assignment to '3'. Now also we avoid column 'D', minimum number is in D.
- Next we go to the row 'D'. In which there are two elements 9 and 5 out of which 5 is a smallest number. Therefore we give the assignment to '5', we avoid column 'E', minimum number is in E.
- Now next we go to the row 'E'. In which only one element 8 is remaining, therefore we give the assignment to '8' ,8 is in column C.
- Lastly we go to the row 'C' and give the assignment to '8' in the 1st ly avoided column.

In this way we obtain the Hamiltonian cycle is A-B-D-E-C-A with minimum weight = 2+3+5+8+8=26

It is equivalent to find shortest path from city A to A crossing all cities once is A-B-D-E-C-A with shortest distance is 26 km.

	A	B	C	D	E
A	0	2	8	4	6
B	2	0	7	3	6
C	8	7	0	9	8
D	4	3	9	0	5
E	6	6	8	5	9

5.5.1 Applications Travelling Salesman Problem

The Travelling Salesman Problem (TSP) has numerous applications in various fields. Some of the common applications of TSP are:

Logistics and transportation: The TSP is used to optimize the routing of delivery trucks, sales representatives, and service technicians. By finding the shortest route that visits all necessary locations, TSP can help minimize travel time and reduce transportation costs.

Circuit design: In the field of electronics, TSP is used to optimize the routing of electrical signals in a printed circuit board, which is a critical step in the design of electronic circuits.

DNA sequencing: In bioinformatics, TSP is used to find the shortest path to sequence a DNA molecule. DNA sequencing involves finding the order of nucleotides in a DNA molecule, which can be modeled as a TSP.

Network design: TSP is used to optimize the design of communication networks, such as the placement of cell towers, routing of fiber-optic cables, and design of wireless mesh networks.

Robotics: TSP can be applied in robotics to optimize the path planning of robots to visit a set of locations in the most efficient manner.

These are just a few examples of the many applications of TSP. The problem is widely studied and has numerous practical applications in various fields.

5.6 Comparison Between Eulerian and Hamiltonian Graph

Hamiltonian Graph	Eulerian Graph
Covers all the vertices of a graph exactly once.	Travers all the edges of a graph exactly once.
It can contain same edge mutiple times.	It can contain same vertices mutiple times.
Every pair of distinct non-adjacent vertex the addition of the sum of their degrees and the length of the shortest path for that pair is greater than or equal to the number of vertices.	The graph should have at most two odd degree vertices.
No nodes may be omitted.	No edges may be omitted.
The vertices do not need to be of even degree.	All vertices have to be even degree.

Chapter 6

Applications of Eulerian and Hamiltonian Graph

Eulerian and Hamiltonian graph is one of the topic pf graph theory which has wide application in other area of mathematics as well as in other branches of science. It also plays significant role in our everyday life. Application of Eulerian and Hamiltonian graph in our day to day life and various fields of science.

Applications of Eulerian and Hamiltonian graph in Everyday Life

6.1 GPS or Google Maps

GPS or Google Maps are to find a shortest route from one destination to another. The destinations are Vertices and their connections are Edges consisting distance. The optimal route is determined by the software. Schools / Colleges are also using this technique to pick up students from their stop to school. Each stop is a vertex and the route is an edge. A Hamiltonian path represents the efficiency of including very vertex in the route.

6.2 To clear road blockage

When roads of a city are blocked due to ice.Planning is needed to put salt on the roads. Then Euler paths or circuits are used to traverse the streets in the most efficient way.

Application of Eulerian and Hamiltonian in Biology

6.3 DNA fragment assembly

DNA (deoxyribonucleic acid) is found in every living organism and is a storage medium for genetic information. A DNA strand is composed of bases which are denoted by A (adenine), C (cytosine), G (Guanine) and T (thymine). The familiar DNA double helix arises by the bondage of two separate strands with the Watson-Crick complementarity (A and T are complementary; C and G are complementary) leading to the formation of such double strands.

DNA sequencing and fragment assembly is the problem of reconstructing full strands of DNA based on the pieces of data recorded. It is of interest to note that ideas from graph theory, especially Eulerian circuits have been used in a recently proposed approach to the problem of DNA fragment assembly.

6.4 Usage in Pandemic Situation

Application of Eulerian and Hamiltonian graph in Vaccination

Vaccination is a simple, easy, and effective way of protecting people from harmful diseases before it come into contact with us. Vaccination is done in the hospitals and in covid-19 vaccination two doses is given to a single person i.e. first dose and second dose after 84 days. If we apply the divided the total hospitals of a particular city into 5 parts and apply EULERIAN and HAMILTONIAN graph in each of it, then we can easily increase the speed of vaccination. Suppose we have total "W" hospitals in particular city, as shown in fig 6.1

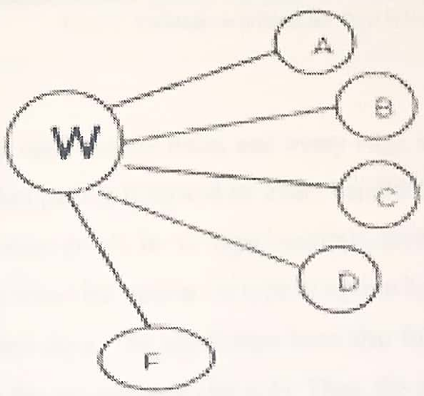


Fig 6.1. Categorization of the hospitals w.r.to age limit

Then we divide those W hospitals in 5 parts (on the basis that the route/path of thehospitals are different)- 'A','B','C','D','E'.

In 'A' Type hospitals (fig 6.2), first dose of vaccine will be provided to 45+ candidates. The candidates are requested to follow Eulerian and Hamiltonian path in order to receive first dose. As shown in the fig 6.2

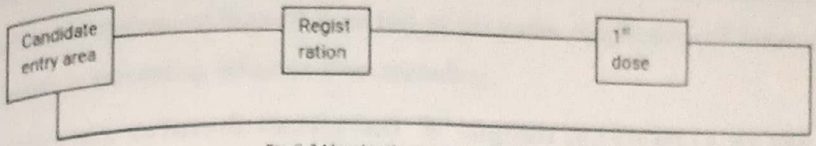


Fig 6.2 Vaccination process for Type A hospitals

As we able to see that every point or vertex is covered exactly once and every edge or path from one point to another is also covered exactly once hence the path followed is Eulerian and Hamiltonian path and thus the graph formed is by applying the Eulerian and Hamiltonian graph. Similarly, in 'B' type hospitals (fig 6.3), first dose of vaccine will be provided to 18+ can didates. The candidates are requested to follow Eulerian and Hamiltonian path in order to receive first dose.

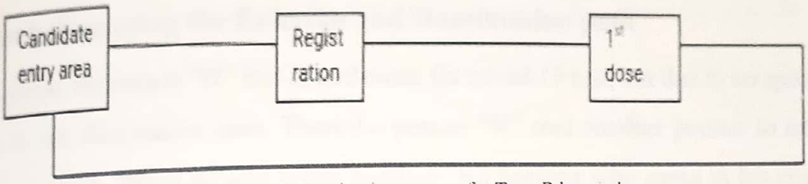


Fig 6.3 Vaccination process for Type B hospitals

Every point or vertex is used exactly once, and every edge or path is also used once hence Eulerian and Hamiltonian path is followed by every candidate and thus the graph so formed is Eulerian and Hamiltonian graph. In 'C' type hospitals, seconddose of vaccine will be provided and the candidate who visit option 1st type or option 2nd type hospitals, will visit here after 84 days of their first dose. The candidates here also follow Eulerian and Hamiltonian path in order to receive the second dose (fig 6.4). Thus, the graph so formed is Eulerian and Hamiltonian graph.

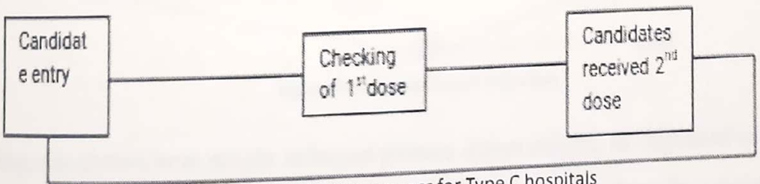


Fig 6.4. Vaccination process for Type C hospitals

In 'D' type hospitals, covid-19 +ive patients are treated. The candidate visit to this hospital may or may not visit to 'A' or 'B' type hospital. In 'E' type hospitals, covid-19 test should be done. The candidate whose report will be positive should refer to 'D' type hospital and also should advise to disconnect himself from rest of the paths. Application of Euler and Hamiltonian graph in preventing infection from spreading.

In above text, we have divided the total 'W' hospitals into 4 types i.e. we stop visiting of random candidate into any hospitals for vaccination of 1st and 2nd dose, for covid-19 treatment and for covid-19 test. By disconnecting the path (and providing Eu- lerian and Hamiltonian path for specific need) of candidate for vaccination from the candidate for covid-19 treatment not only help people for easily vaccinated but also prevent people from coming in contact with infected person.

By applying the Eulerian and Hamiltonian path in each type hospitals, also secure/ prevent people from coming in contact with each other.

Condition before using the Eulerian and Hamiltonian path

Suppose the person "N" feel ill and come for covid-19 test, but due to no specific path, he went in the vaccination area. There the person "N" met another person to ask for the covid-19 test area. When he was tested positive. All persons who came in his contact also tested positive and this process remain continue, as depicted in fig 6.5.

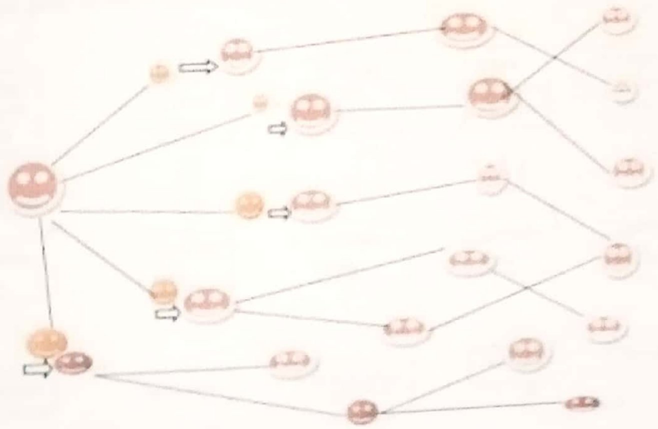


Figure 6.5. Spreading of infection

The above diagram shows how single infected person infect others, as depicted in fig 6.5 . This spreading of infection can be prevented if we provide the separate place for covid-19 test. The best way to stop spreading of covid-19 in hospitals, vaccina-tion center and tested area is to disconnect their path from each other. This can be achieved by simplifying disconnecting some paths and by follow the Eulerian and Hamiltonian path in each hospital.

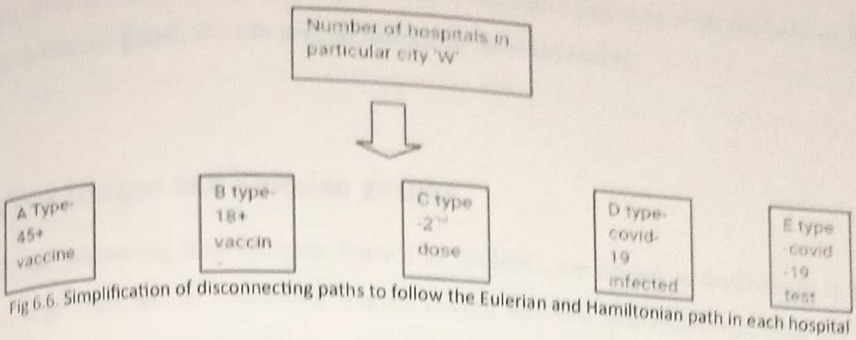


Fig 6.6. Simplification of disconnecting paths to follow the Eulerian and Hamiltonian path in each hospital

Condition after using the Eulerian and Hamiltonian path

Suppose the person "N" feel ill and come for covid-19 test, as it is specified that testing for covid-19 is in hospital of type "D". The person "N" will visit there and no one else get infected from him in case he will tested positive. By disconnecting five paths from each other, we can easily safe millions of live. In other words, we can say by dividing the single root vertex into 5 other vertex and by dividing single edge into five different edges, we can prevent infection from spreading. Each type of hospital follows the Eulerian and Hamiltonian path i.e. no person is allowed to repeat same room again and to skip any room or process (no vertex repeat itself and every vertex visit once, every edge is in path exactly once). Now, to understand Eulerian and Ham- iltonian path in each type of hospitals.

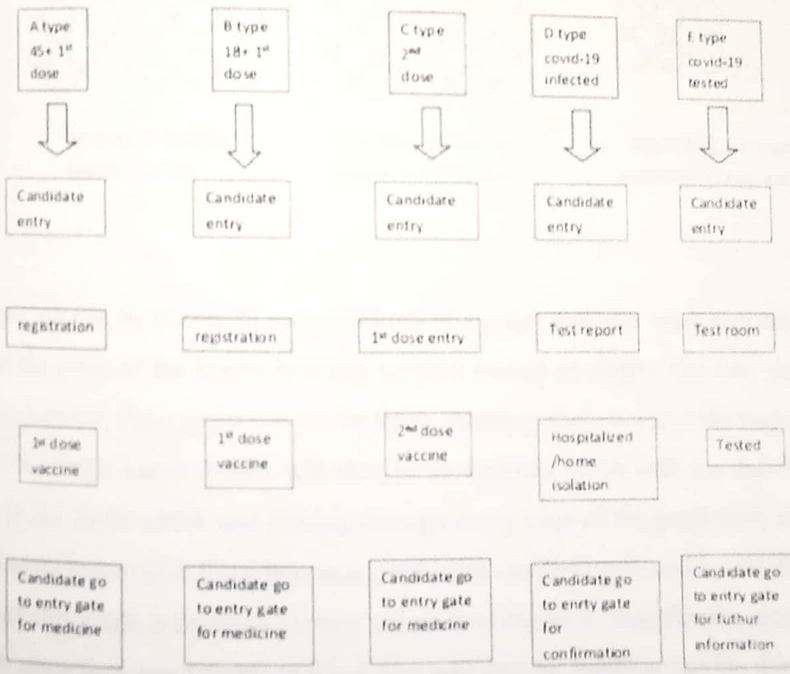


Fig 6.7. Preventing infection to spread using Eulerian and Hamiltonian graph

Clearly from the above graph (fig 6.7) we can understand that how with the help of Eulerian and Hamiltonian graph, we can prevent infection from spreading.

6.5 Floor Designs and Eulerian graphs

Traditional interesting floor designs, known as "kolam", are drawn as decorations in the floor and in large sizes and in interesting shapes, during festivals and weddings, with the drawing done with rice flour or rice paste especially in South India.

Generally, in drawing a kolam, first a suitable arrangement of dots is made and then lines going around the dots are drawn. An example kolam is shown in Fig 6.8(a) where the dots are ignored. One of the types of kolam, is known as 'kambi kolam' (the Tamil word kambi meaning wire). In this type, the kolam is made of one or more of such 'kambi's . The kolam in Fig 6.8(a) is of this type and is made of a single 'kambi' whereas the kolam in Fig 6.8(c) is made of three 'kambi's.

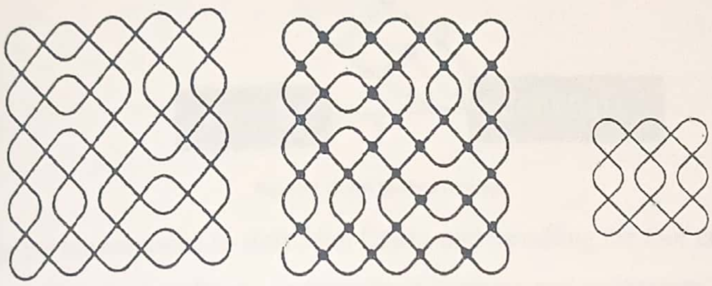


Fig 6.8(a): Single Kambi kolam

Fig 6.8(b) graph of kolam in fig 6.8(a)

Fig 6.8(c) a kolam made of three kambis

A kolam drawing can be treated as a special kind of a graph with the crossings considered as vertices and the parts of the kambi between vertices treated as edges. The only restriction is that unlike in a graph, these edges can not be freely drawn as there is a specific way of drawing the kolam. The single kambi kolam will then be an Eulerian graph with the drawing starting and ending in the same vertex and passing through every edge of the graph only once. In Fig 6.8(b), the kambi kolam of is Fig 6.8(a) as a graph with vertices (indicated by small thick dots) and edges and this graph is Eulerian asevery vertex is of degree 4. Note that the graph of kambi kolam (with more than one kambi) in Fig 6.8(c) will also be Eulerian but the drawing of the kolam in the way it will be done by the kolam practitioners will not be giving rise to the Eulerian circuit. On the other hand in the case of single kambi kolam it is of interest to note that the drawing of the kolam will give a tracing of an Eulerian circuit in the corresponding graph.

6.6 Airplane Route Problem

Suppose, a businessman have to visit in four cities of Bangladesh for his business purpose. He prefers air routes because of short time arrival. He also has an assignment for making a list of rating of services of the six air service providers in those regions. Air services in these regions have a form of following routes in the graph below:

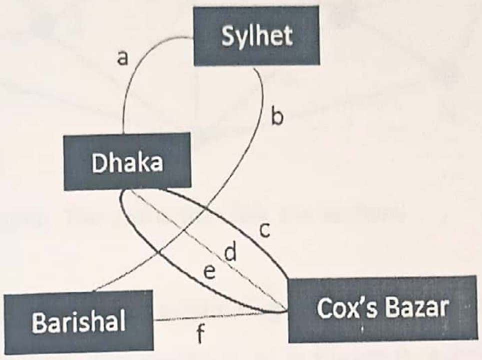


Figure 9. Graph Routing Problem

Is it possible for the businessman to start from Dhaka and travelling the four cities with the six airways and then return back to Dhaka in purpose of business and assignment?

Solution

Here, there are four places to travel or four vertices. Now we will check for the vertices that the Euler theorem is satisfied or not.

For Dhaka, Degree is 4, which is even.

For Sylhet, Degree is 2, which is even.

For Barisal, Degree is 2, which is even.

For Cox's Bazar, Degree is 2, which is even.

That is, all the four vertices has even degree. So by Euler theorem is satisfied and hence it is possible for the businessman to start from Dhaka and travelling the four cities with the six airways and then return back to Dhaka in purpose of business and assignment.

6.7 Organizing Transpiration Networks

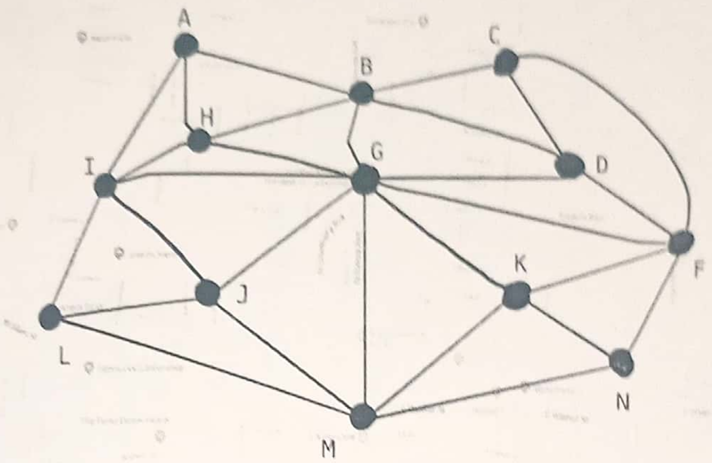


Fig 6.10(a) The paths that link the stations

We can use an Eulerian graph for organizing transpiration networks that can use par-ticular paths just only one time. For example, as in Figure A, a postal carrier wants to use bus stations (vertices) to traverse each street (edges) exactly once. Besides, he has to start and end his route at the same station. To let him do that, the graph must be an Euler circuit. We start checking the degrees of the vertices one by one. As soon as we caught an odd vertex, we know that the graph will be not an Euler circuit. The graph (Figure A) has eight vertices A, B, C, F, N, M, L,and I; their degrees are odds. So it is not an Euler circuit. Let's try to make the graph an Eulerian circuit. There are actually many several Euler circuits he could have taken. One of them could be by deleting four edges AI, BC, FN, and LM (Figure A). Then,all vertices will have even degrees. Now, the mail carrier can traverses each street exactly once, and he can start and end from one station.